Exact results in the gauge/gravity duality

Lorenzen, Jakob Christian

Awarding institution:
King's College London

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Jakob Christian Lorenzen

Department of Mathematics, King's College London,
The Strand, London WC2R 2LS, United Kingdom

Thesis supervisor: Prof. Dario Martelli

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Abstract

In this thesis we study the gauge/gravity duality and exact results in supersymmetric quantum field theories obtained using localization. We construct the gravity duals to a broad class of $\mathcal{N} = 2$ supersymmetric gauge theories defined on a general class of three-manifolds. The gravity backgrounds are supersymmetric solutions of gauged four-dimensional supergravity and encompass all known examples of such solutions. We find that the holographically renormalized on-shell action agrees with the free energy of the field theory, which has previously been computed via localization of the partition function. Next, we study the Casimir energy of four-dimensional $\mathcal{N} = 1$ supersymmetric field theories in the context of the rigid limit of new minimal supergravity. We revisit the computation of the localized partition function on $S^1 \times S^3$, and consider the same theories in the Hamiltonian formalism on $\mathbb{R} \times S^3$. We compute the vacuum expectation value of the canonical Hamiltonian using zeta function regularization, and show that this interpolates between the *supersymmetric Casimir energy* and the ordinary Casimir energy of supersymmetric field theories. In general, the Casimir energy depends on the regularization scheme and is therefore *ambiguous*. However, we show that for $\mathcal{N} = 1$ supersymmetric field theories on the cylinder $\mathbb{R} \times S^3$, the supersymmetric Casimir energy is well-defined and scheme-independent, provided the regularization scheme preserves supersymmetry. Finally, we investigate the gravity duals of such $\mathcal{N} = 1$ theories on $\mathbb{R} \times S^3$. Specifically, we study supersymmetric solutions of five-dimensional minimal gauged supergravity using a known classification. We propose an ansatz based on a four-dimensional local *orthotoric* Kähler metric and reduce the problem to a single sixth-order equation for two functions, each of one variable. We find an analytic, asymptotically locally AdS solution comprising five parameters. For a conformally flat boundary, this reduces to a previously known solution with three parameters, representing the most general solution of this type known in minimal gauged supergravity. We discuss the relevance for this solution to account for the supersymmetric Casimir energy, finding the answer to be in the negative.

Contents

List of Figures

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Contents of this thesis

This thesis presents work published in [1–4], and is organized as follows. We begin in chapter 1 with an overview of supersymmetric field theories on curved backgrounds, localization, and the gauge/gravity duality, anticipating along the way several results of the thesis. In chapter 2 we construct the gravity duals of supersymmetric gauge theories on three-manifolds, based on [1]. Chapter 3 concerns the computation and proof of the scheme-independence of the supersymmetric Casimir energy of $\mathcal{N} = 1$ field theories on $\mathbb{R} \times S^3$. This based on [2] and [3]. In chapter 4, we construct supersymmetric solutions of five-dimensional minimal gauged supergravity. We investigate whether these solutions can holographically account for the supersymmetric Casimir energy, finding the answer to be in the negative. This chapter is based on [4]. Chapter 5 contains concluding remarks. In addition, several appendices are included from the above references.

Chapter 1

Introduction and summary

Quantum field theory is the framework of modern particle physics. At weak coupling, perturbation theory in terms of Feynman diagrams has been a powerful technique, however, this approach cannot capture the full dynamics of quantum field theory.

A remarkable tool for obtaining exact results at strong coupling and large N has been the gauge/gravity duality. This is the conjecture that certain quantum field theories have a dual description in terms of gravity, more precisely string theory or M-theory. The example of the original conjecture [5] was four-dimensional $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) dual to type IIB string theory on an $AdS_5 \times S^5$ background. Since $\mathcal{N} = 4$ SYM is not only maximally supersymmetric, but also a conformal field theory (CFT), the duality is also known as the AdS/CFT correspondence. Although there is no mathematical proof of the conjecture, there is by now much evidence including various settings beyond the original example.

On the field theory side, one technique that has been studied extensively for $\mathcal{N} = 4$ SYM is *integrability* (see [6] and references therein). In the *planar* limit, where the number of colours N goes to infinity, while the coupling constant g_{YM} goes to zero in such a way that the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ remains finite, the large amount of symmetry in the theory allows for exact computations. This has allowed for checks of the gauge/gravity duality.

A more recent computational technique, which will be of interest in this thesis, is supersymmetric *localization*. For supersymmetric field theories defined on compact Riemannian manifolds, under appropriate circumstances it can be shown that path integrals localize in field space. This reduces the infinite-dimensional path integral to a finite-dimensional matrix integral, in many instances allowing the integral to be computed exactly. Results obtained via localization are valid for any value of the coupling, and in the strong-coupling limit these serve as checks or predictions for results obtained from the gauge/gravity duality. Indeed, this has led to novel examples of the gauge/gravity duality, where the field theory is defined on a non-trivially curved background.

In the remainder of this chapter, we review more details of gauge/gravity duality, localization, and the interplay between these techniques.

1.1 Gauge/gravity duality

According to the gauge/gravity duality, some quantum field theories are equivalent to string theory or M-theory on certain backgrounds. In particular, taking in the string theory the limit where the string coupling and α' go to zero, the string theory is described by classical supergravity. In the field theory this corresponds to the limit of large N and large 't Hooft coupling λ.

The first example of a gauge/gravity duality was conjectured in [5], with further details in [7, 8]. The conjecture states that four-dimensional $SU(N)$ $\mathcal{N}=4$ SYM theory is equivalent to type IIB string theory on an $AdS_5 \times S^5$ background. The motivation for such a remarkable conjecture comes from brane constructions in string theory. Type IIB string theory contains closed strings, as well as open strings whose end points are restricted to so-called Dp-branes. These are hypersurfaces extending in p spacelike directions. The excitations of an open string give rise to a gauge theory living on the $(p+1)$-dimensional world volume of the brane. In particular, a stack of coincident Dp-branes gives rise to a non-Abelian gauge group, and the gauge theory inherits supersymmetry from the string theory. On the other hand, in an appropriate limit, Dp-branes also have a description as solutions of the equations of motion of ten-dimensional classical supergravity.

The conjecture of [5] is motivated specifically from considering a stack of N coincident D3-branes. The gauge theory on the four-dimensional world volume of the branes is $\mathcal{N}=4$ SYM with an $SU(N)$ gauge group, while the $AdS_5 \times S^5$ spacetime arises as the near-horizon geometry of the branes in the supergravity solution. In terms of the duality, the field theory is said to live on the conformal boundary of AdS$_5$, with the isometry group $SO(2,4)$ acting on the four-dimensional boundary as the conformal group. Indeed $\mathcal{N}=4$ SYM is a conformal field theory with vanishing β-function. The isometry group $SO(6) \simeq SU(4)$ of the S^5 corresponds to the R-symmetry of the field theory.

Another concrete example of the gauge/gravity duality was conjectured in [9]. This work followed the construction of a three-dimensional $\mathcal{N}=8$ superconformal theory constructed by Bagger and Lambert [10, 11] (see also [12]), which was conjectured to be related to a specific theory of M2-branes for the lowest Chern-Simons levels [13, 14]. The gauge theory of [9] is a three-dimensional $\mathcal{N}=6$ superconformal Chern-Simons-matter theory, known as ABJM theory. It contains two copies of $U(N)$ Chern-Simons theory with opposite levels, k and $-k$, coupled to four matter supermultiplets in the bifundamental representation of the gauge group $U(N)_k \times U(N)_{-k}$. From its M2-brane origin, its gravity dual was conjectured to be $AdS_4 \times S^7/\mathbb{Z}_k$ in eleven-dimensional supergravity.

A central prescription in the gauge/gravity duality is the identification of the partition function of the gauge theory with that of the string/M-theory. In particular, in the limit where the string/M-theory is well approximated by classical supergravity, the gravity partition function will be dominated by the on-shell field configurations in a saddle point approximation,

$$\mathrm{e}^{-S_{\text{sugra}}[M]} \;=\; Z_{\text{CFT}}[\partial\mathcal{M}] \; . \tag{1.1}$$

Here, M is the bulk manifold with conformal boundary ∂M, and S_{sugra} is formally the supergravity action evaluated on-shell. From the supergravity theory, one may then holographically compute field theory quantities, e.g correlation functions. Of particular interest in Chapter 2 will be free energy of the field theory, which we define as $\mathcal{F} = -\log Z_{\text{CFT}}$. From (1.1), we see that the free energy at large N is formally just the on-shell gravity action. However, in general, S_{sugra} is a divergent quantity involving an integral over the infinite volume of an AdS space. These divergences can be removed using the technique of *holographic renormalization*, as we will see concretely in section 2.3 below.

The gauge/gravity duality has also been applied to settings quite different from the original conjecture, for example providing new insights in condensed matter physics. Holographic superconductors have been constructed as AdS black holes in theories with a Maxwell field and a charged scalar field [15, 16]. Note that this case does not rely on supersymmetry. Gravity duals have also been constructed for condensed matter systems displaying scale-invariance, but not Lorentz-invariance [17–19].

In this thesis, we will be interested in supersymmetric quantum field theories defined on curved backgrounds, and their gravity duals. In particular, the motivation to study this setting comes from new exact computations in field theory from localization, requiring a supersymmetric field theory to be defined on a compact Riemannian manifold. We will return to the gauge/gravity duality below, after some discussion of supersymmetry on curved backgrounds and localization.

1.2 Supersymmetric field theories on curved spaces

Results obtained using localization motivated the study of how to construct field theories with rigid supersymmetry on curved backgrounds. A theory can be placed on a curved background by minimally coupling it to the metric, but in general the theory will then no longer be supersymmetric. A more systematic approach for obtaining theories with rigid supersymmetry was started in [20] in four dimensions. It was then developed further for the three-dimensional case in [21–24]. We shall now review this formalism in four and three dimensions, respectively.

1.2.1 Four dimensions

In four dimensions, one may obtain $\mathcal{N} = 1$ supersymmetric field theories with an R-symmetry on curved backgrounds [20] from so-called off-shell "new minimal supergravity" [25]. We give here an overview in Euclidean signature following [26], while in section 3.4 below we will analytically continue this formalism to Lorentzian signature.

The gravity multiplet of new minimal supergravity contains the metric, a gravitino ψ_μ, an auxiliary two-form $B_{\mu\nu}$, and an auxiliary vector field A_μ which is a gauge field for the local chiral symmetry. Taking a rigid limit by appropriately sending the Plank mass to infinity, one obtains a rigid supersymmetric theory containing vector and chiral multiplets, while the gravity multiplet fields remain as non-dynamical background fields. Rather than the two-form $B_{\mu\nu}$, we will work with the one-form $V = *_4 \mathrm{d}B$, where we denote by $*_d$ the Hodge dual in d dimensions. From its definition, V is conserved, $\nabla_\mu V^\mu = 0$. In Euclidean signature, A_μ and V_μ may take complex values, while we restrict the metric to be real. The real part of A_μ transforms locally as a gauge field and couples to the $U(1)_R$ R-symmetry, while the imaginary part must be a well-defined one-form. To obtain a rigid supersymmetric theory, it is necessary that the background admits a solution ζ or $\widetilde{\zeta}$ to at least one of the Killing spinor equations,

$$
\begin{aligned}
(\nabla_\mu - \mathrm{i}A_\mu)\zeta + \mathrm{i}V_\mu\zeta + \mathrm{i}V^\mu\sigma_{\mu\nu}\zeta &= 0 \\
(\nabla_\mu + \mathrm{i}A_\mu)\widetilde{\zeta} - \mathrm{i}V_\mu\widetilde{\zeta} - \mathrm{i}V^\mu\widetilde{\sigma}_{\mu\nu}\widetilde{\zeta} &= 0 \ .
\end{aligned} \tag{1.2}
$$

We follow here the conventions of [26]. The 2×2 matrices σ_μ and $\widetilde{\sigma}_\mu$ generate the Clifford algebra Cliff$(4, 0)$, and the spinors ζ and $\widetilde{\zeta}$ are two-component complex spinors of opposite chirality and with opposite charge under the gauge field A_μ, which couples to the R-symmetry. In Lorentzian signature, $\widetilde{\zeta}$ would be the complex conjugate of ζ, but in Euclidean signature the number of degrees of freedom is doubled by allowing the two spinors to be independent.

Is it then natural to ask which manifolds admit solutions to (1.2). In Euclidean signature one can construct from the spinors ζ and $\widetilde{\zeta}$ the almost complex structures,

$$
J^\mu{}_\nu = \frac{2\mathrm{i}}{|\zeta|^2}\zeta^\dagger\sigma^\mu{}_\nu\zeta \ , \qquad \widetilde{J}^\mu{}_\nu = \frac{2\mathrm{i}}{|\widetilde{\zeta}|^2}\widetilde{\zeta}^\dagger\widetilde{\sigma}^\mu{}_\nu\widetilde{\zeta} \ . \tag{1.3}
$$

A necessary and sufficient condition for a four-dimensional Riemannian manifold to admit a solution to (1.2) is that at least one of the almost complex structures is integrable [23, 27]. When there exists non-zero solutions ζ and $\widetilde{\zeta}$ to both equations (1.2), the spinor bilinear

$$
K^\mu = \zeta\sigma^\mu\widetilde{\zeta} \ , \tag{1.4}
$$

is a complex Killing vector, hence comprising two real Killing vectors. Moreover, it is holomorphic with respect to both complex structures (1.3). In Lorentzian signature, equations (1.2) admit a solution if and only if the background admits a null Killing vector [28]. This approach of constructing spinor bilinears has previously been employed in other contexts to determine geometric restrictions imposed by supersymmetry. In particular, this has led to classifications of supersymmetric solutions of supergravity in terms of G-structures [29–31], and in chapter 4 we will use the conditions derived in [32] for a solution of five-dimensional minimal gauged supergravity to preserve supersymmetry.

The vector fields A_μ and V_μ are only defined up to shifts parametrized by a vector U_μ,

$$A_\mu \;\to\; A_\mu + \frac{3}{2}U_\mu \;, \qquad V_\mu \;\to\; V_\mu + U_\mu \;, \tag{1.5}$$

provided U_μ is holomorphic, namely $J^\mu{}_\nu U^\nu = \mathrm{i}U^\mu$, and divergenceless $\nabla_\mu U^\mu = 0$. When the Killing vector K commutes with its complex conjugate, $K^\nu\nabla_\nu\bar{K}^\mu - \bar{K}^\nu\nabla_\nu K^\mu = 0$, then U^μ must in fact be of the form $U^\mu = \kappa K^\mu$, where κ is a complex function satisfying $K^\mu\partial_\mu\kappa = 0$. In chapter 3 we will take κ to be a constant. Note that the combination $A^{\mathrm{cs}}_\mu \equiv A_\mu - \frac{3}{2}V_\mu$ is independent of the choice of U_μ.

The following Lagrangian was presented in [26] for an $\mathcal{N}=1$ vector multiplet containing a gauge field A_μ, a pair of two-component spinors $\lambda, \widetilde{\lambda}$ of opposite chirality, and an auxiliary scalar field D,

$$\mathcal{L}_{\mathrm{vector}} \;=\; \mathrm{Tr}\left[\frac{1}{4}\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu} + \frac{\mathrm{i}}{2}\lambda\sigma^\mu D^{\mathrm{cs}}_\mu\widetilde{\lambda} + \frac{\mathrm{i}}{2}\widetilde{\lambda}\widetilde{\sigma}^\mu D^{\mathrm{cs}}_\mu\lambda - \frac{1}{2}D^2\right] \;, \tag{1.6}$$

where $\mathsf{F}_{\mu\nu} = \partial_\mu\mathsf{A}_\nu - \partial_\nu\mathsf{A}_\mu - \mathrm{i}[\mathsf{A}_\mu, \mathsf{A}_\nu]$ and $D^{\mathrm{cs}}_\mu = \nabla_\mu - \mathrm{i}\mathsf{A}_\mu - \mathrm{i}q_R A^{\mathrm{cs}}_\mu$ with q_R the R-charge of the field. The fields $\{\mathsf{A}_\mu, \lambda, \widetilde{\lambda}, D\}$ have R-charges $\{0, 1, -1, 0\}$ and all transform in the adjoint of the gauge group G. The vector multiplet Lagrangian (1.6) is invariant under the supersymmetry transformations

$$\begin{aligned}
\delta\mathsf{A}_\mu &= \mathrm{i}\zeta\sigma_\mu\widetilde{\lambda} + \mathrm{i}\widetilde{\zeta}\widetilde{\sigma}_\mu\lambda \\
\delta\lambda &= \mathsf{F}_{\mu\nu}\sigma^{\mu\nu}\zeta + \mathrm{i}D\zeta \\
\delta\widetilde{\lambda} &= \mathsf{F}_{\mu\nu}\widetilde{\sigma}^{\mu\nu}\widetilde{\zeta} - \mathrm{i}D\widetilde{\zeta} \\
\delta D &= -\zeta\sigma^\mu\Big(D_\mu\widetilde{\lambda} - \frac{3\mathrm{i}}{2}V_\mu\widetilde{\lambda}\Big) + \widetilde{\lambda}\widetilde{\sigma}^\mu\Big(D_\mu\lambda + \frac{3\mathrm{i}}{2}V_\mu\lambda\Big) \;,
\end{aligned} \tag{1.7}$$

with $D_\mu = \nabla_\mu - \mathrm{i}\mathsf{A}_\mu - \mathrm{i}q_R A_\mu$. In Euclidean signature, the tilded fields are independent of the untilded. When turning to Lorentzian signature, these will be related by conjugation. Crucially for the localization argument of [26], it was shown therein that the Lagrangian (1.6) is itself a total supersymmetry variation.

Likewise, the chiral multiplet in [26] was also shown to be a total supersymmetry variation. In fact, it is a sum of four such variations,

$$\mathcal{L}_{\mathrm{chiral}} \;=\; \delta_\zeta V_1 + \delta_\zeta V_2 + \delta_\zeta V_3 + \delta_\zeta V_U . \tag{1.8}$$

A chiral multiplet contains two complex scalars $\phi, \widetilde{\phi}$, a pair of two-component complex spinors $\psi, \widetilde{\psi}$ of opposite chirality, and two complex auxiliary fields $F, \widetilde{F}$. These fields $\{\phi, \psi, F, \widetilde{\phi}, \widetilde{\psi}, \widetilde{F}\}$ have R-charges $\{r, r-1, r-2, -r, -r+1, -r+2\}$. The untilded fields transform in a representation $\mathcal{R}$ of the gauge group, while the tilded transform in the conjugate representation $\mathcal{R}^*$. Again, in Euclidean signature the tilded and untilded fields are independent. In components, the Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\text{chiral}} \quad = \quad & D_\mu \widetilde{\phi} D^\mu \phi + \mathrm{i} V^\mu \big(D_\mu \widetilde{\phi}\, \phi - \widetilde{\phi} D_\mu \phi \big) + \frac{r}{4} \big(R + 6 V_\mu V^\mu \big) \widetilde{\phi} \phi + \widetilde{\phi} D \phi - \widetilde{F} F \\ & + \mathrm{i} \widetilde{\psi} \widetilde{\sigma}^\mu D_\mu \psi + \frac{1}{2} V^\mu \widetilde{\psi} \widetilde{\sigma}_\mu \psi + \mathrm{i} \sqrt{2} \big(\widetilde{\phi} \lambda \psi - \widetilde{\psi} \widetilde{\lambda} \phi \big) \ , \end{aligned} \tag{1.9}$$

where R is the Ricci scalar of the background metric. The Lagrangian (1.9) is invariant under the supersymmetry transformations

$$\begin{aligned} \delta \phi &= \sqrt{2} \zeta \psi \\ \delta \psi &= \sqrt{2} F \zeta + \mathrm{i} \sqrt{2} (\sigma^\mu \widetilde{\zeta}) D_\mu \phi \\ \delta F &= \mathrm{i} \sqrt{2} \widetilde{\zeta} \widetilde{\sigma}^\mu \left(D_\mu \psi - \frac{\mathrm{i}}{2} V_\mu \psi \right) - 2\mathrm{i} (\widetilde{\zeta} \widetilde{\lambda}) \phi \\ \delta \widetilde{\phi} &= \sqrt{2} \widetilde{\zeta} \widetilde{\psi} \\ \delta \widetilde{\psi} &= \sqrt{2} \widetilde{F} \widetilde{\zeta} + \mathrm{i} \sqrt{2} (\widetilde{\sigma}^\mu \zeta) D_\mu \widetilde{\phi} \\ \delta \widetilde{F} &= \mathrm{i} \sqrt{2} \zeta \sigma^\mu \left(D_\mu \widetilde{\psi} + \frac{\mathrm{i}}{2} V_\mu \widetilde{\psi} \right) + 2\mathrm{i} \widetilde{\phi} (\zeta \lambda) \ . \end{aligned} \tag{1.10}$$

One may couple to the theory an arbitrary number of chiral multiplets, each with different R-charge r_I, and also add a superpotential [20]. We will return to the Lagrangian (1.9) in chapter 3.

1.2.2 Three dimensions

The analysis of [20] was further developed in [21–24] for theories on Riemannian three-manifolds. In particular, [24] constructed $\mathcal{N} = 2$ supersymmetric gauge theories with an R-symmetry on Riemannian three-manifolds, encompassing all previously known examples. Although only a linearized formulation is known for new minimal supergravity in three dimensions, it was argued in [24] that this is sufficient up to terms that vanish when taking the rigid limit.

The gravity multiplet contains the metric $g_{\mu\nu}$, two gravitini $\psi_\mu^{(3)}, \widetilde{\psi}_\mu^{(3)}$, a two-form gauge field $B_{\mu\nu}^{(3)}$, and two gauge fields[1] $A_\mu^{(3)}$ and C_μ. While we take the metric to be real, the gauge field may be complex. We will use the dual fields, the vector $V^{(3)} = -\mathrm{i} *_3 \mathrm{d} C$ and the scalar field $h = *_3 \mathrm{d} B^{(3)}$, from which it follows that $\nabla^\mu V_\mu^{(3)} = 0$. Again these remain as background fields. The resulting theory possesses rigid

[1]The superscript $^{(3)}$ is intended to distinguish the three-dimensional fields from the four-dimensional ones.

supersymmetry if and only if the gravitini vanish and there exists a spinor χ or $\widetilde{\chi}$ solving one of the equations

$$\begin{aligned}\left(\nabla_\mu - \mathrm{i}A^{(3)}_\mu\right)\chi + \frac{1}{2}\mathrm{i}h\gamma_\mu\chi + \mathrm{i}V^{(3)}_\mu\chi + \frac{1}{2}\epsilon_\mu{}^{\nu\rho}V^{(3)}_\nu\gamma_\rho\chi &= 0 \\ \left(\nabla_\mu + \mathrm{i}A^{(3)}_\mu\right)\widetilde{\chi} + \frac{1}{2}\mathrm{i}h\gamma_\mu\widetilde{\chi} - \mathrm{i}V^{(3)}_\mu\widetilde{\chi} - \frac{1}{2}\epsilon_\mu{}^{\nu\rho}V^{(3)}_\nu\gamma_\rho\widetilde{\chi} &= 0\ ,\end{aligned} \tag{1.11}$$

where γ_a are the Pauli matrices generating the Clifford algebra Cliff(3,0) in an orthonormal frame.

When $A^{(3)}_\mu$, $V^{(3)}_\mu$, and h are real, and χ solves the upper equations in (1.11), the lower equation in (1.11) is solved by its charge conjugate χ^c. This is the set-up of the localization computation performed in [33], which we shall discuss further in the next section.

From the spinor χ we can then construct a Killing vector $K^{(3)}$, and choosing appropriately the coordinate ψ, this is given by

$$K^{(3)} \equiv \chi^\dagger\gamma^\mu\chi\,\partial_\mu = \partial_\psi\ . \tag{1.12}$$

This vector defines a transversely holomorphic foliation of the three-manifold. Introducing a local complex coordinate z, the metric is given in terms of the functions $\Omega(z,\bar{z})$ and $c(z,\bar{z})$, and the one-form $a = a(z,\bar{z})\mathrm{d}z + \overline{a(z,\bar{z})}\mathrm{d}\bar{z}$, as

$$\mathrm{d}s^2_3 = \Omega^2\left(\mathrm{d}\psi + a\right)^2 + c^2\mathrm{d}z\mathrm{d}\bar{z}\ . \tag{1.13}$$

Similar to the four-dimensional case, the background fields are defined up to shifts of the form,

$$A^{(3)}_\mu \to A^{(3)}_\mu + \frac{3}{2}\kappa\Omega\eta_\mu \qquad V^{(3)}_\mu \to V^{(3)}_\mu + \kappa\Omega\eta_\mu\ , \qquad h \to h+\kappa\ , \tag{1.14}$$

where

$$\eta = \mathrm{d}\psi + a\ , \tag{1.15}$$

and the function κ must satisfy $K^{(3)}_\mu\partial^\mu\kappa = 0$. The three-manifold admits an *almost contact structure*. The nowhere vanishing one-form η is the *almost contact form* on the three-manifold[2] and $K^{(3)}$ is the associated *Reeb vector field.* These satisfy

$$K^{(3)}\lrcorner\,\eta = 1\ , \qquad K^{(3)}\lrcorner\,\mathrm{d}\eta = 0\ . \tag{1.16}$$

For further details on almost contact structures, see for instance [34] or the appendix of [24].

Langrangians and supersymmetry variations for Chern-Simons multiplets, Yang-

[2]If $\eta\wedge\mathrm{d}\eta$ is nowhere vanishing then η is a *contact form* and the three-manifold has a *contact structure.* This is not necessarily the case in the current context.

Mills multiplets, and chiral multiplets on three-manifolds with the geometry described above were given in [24]. We shall not need them explicitly here. However, the above formulae will be recovered in section 2.2 below, as the background geometry on the boundary of four-dimensional supergravity solutions.

1.3 Localization

Although localization has a longer history [35], the recent interest was sparked by ref. [36], in which the path integral of $\mathcal{N}=2$ super-Yang-Mills theory on the four-sphere was computed. Results obtained via localization have served as non-perturbative tests of conjectured dualities, *e.g.* mirror symmetry in three-dimensional theories [37]. In particular, as these results are valid for any value of the coupling constant, they may serve as checks of the gauge/gravity duality in the strong-coupling limit.

The central point of localization is that under appropriate circumstances, the infinite-dimensional path integral can be reduced to a finite-dimensional integral. Let us consider a supersymmetric quantum field theory defined on a compact Riemannian manifold. Due to the compactness, we assume the field theory is free of infrared divergences. The partition function is defined by the path integral as usual,

$$Z[\phi] \;=\; \int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]} \;, \tag{1.17}$$

where ϕ denotes collectively the fields of the theory. For a supersymmetric theory, we consider a Grassmann-odd supercharge, $\mathcal{Q}$, under which the action is invariant, $\mathcal{Q}S[\phi]=0$. The supercharge squares to a bosonic charge B,

$$\mathcal{Q}^2 \;=\; B \;, \tag{1.18}$$

which may generate a linear combination of spacetime symmetries, global internal symmetries, and gauge symmetries. We assume the integration measure in (1.17) is $\mathcal{Q}$-invariant. Consider then a deformation of the partition function,

$$Z_t[\phi] \;=\; \int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]-t\mathcal{Q}V[\phi]} \;, \tag{1.19}$$

where $V[\phi]$ is a Grassmann-odd operator invariant under B, and t is some parameter. The perturbed partition function is independent of t, which can be shown by differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}t}Z_t[\phi] \;=\; -\int \mathcal{D}\phi\,(\mathcal{Q}V)\mathrm{e}^{-S-t\mathcal{Q}V} \;=\; -\int \mathcal{D}\phi\,\mathcal{Q}\big(V\mathrm{e}^{-S-t\mathcal{Q}V}\big) \;=\; 0 \;. \tag{1.20}$$

In the last equality, we interpreted the integrand as a total derivative on field space, and assumed that there are no boundary terms (or at least that the integral decays sufficiently fast). We can thus choose to compute (1.19) for any convenient value of t. Clearly, for $t = 0$ we recover the original partition function (1.17). Assuming the bosonic part of $\mathcal{Q}V$ is positive semi-definite, we can consider the limit in which $t \to \infty$. In this limit, the integral will be dominated by the locus of field configurations for which

$$\mathcal{Q}V[\phi] \;=\; 0. \tag{1.21}$$

For this reason, $\mathcal{Q}V[\phi]$ is known as the localizing action. In many interesting cases, the path integral localizes to a finite-dimensional integral in this way. Further, one may utilize this localization technique to compute the expectation value of operators by inserting these into the path integral as usual, provided these operator are gauge invariant and BPS. Examples includes vortex loop operators and Wilson loops in three and four dimensions [36, 38–41]. For more detailed reviews on the localization technique, see *e.g.* [42, 43].

Three-manifolds

There has been a number of results in three dimensions. The authors of [44] applied localization to $\mathcal{N} = 3$ superconformal Chern-Simons-matter theories on the round three-sphere. It was shown that the exact partition function and certain Wilson loop observables can be reduced to more manageable matrix models. The authors also wrote down as a matrix integral the partition function Z of the $\mathcal{N} = 6$ ABJM theory [9], which was studied further in [45]. In particular, the authors of [45] showed from the matrix integral that the free energy $\mathcal{F} = -\log Z$ of ABJM theory scales at large N as $N^{3/2}$. This $N^{3/2}$ behaviour had been in need of clarification since it was noticed more than a decade earlier from the study of N coincident M2-branes [46].

The techniques developed in [44] were extended to the partition function of $\mathcal{N} = 2$ theories in [47,48], and to Wilson loops [38,39] and vortex loops [40,41], as mentioned above. Recall the round three-sphere has isometry group $SO(4) \simeq SU(2) \times SU(2)$. The partition function for $\mathcal{N} = 2$ theories on particular squashed three-spheres preserving either $SU(2) \times U(1)$ or $U(1) \times U(1)$ isometry were studied in [49], while a different squashing preserving $SU(2) \times U(1)$ isometry was later studied in [50]. Other topologies have also been considered, such as Lens spaces [51, 52].

In [33], the localized partition function was computed for $\mathcal{N} = 2$ Chern-Simons theories coupled to chiral multiplets, defined on a broad class of Riemannian three-manifolds M_3 with the topology of the three-sphere. The background M_3 preserves a $U(1) \times U(1)$ isometry and encompasses all previous such examples. The field theory is defined on such a background as described in section 1.2.2 above. Recall the Killing spinor solving equations (1.11) gives rise to a Killing vector $K^{(3)}$. If all the orbits of $K^{(3)}$ close, it generates a $U(1)$ isometry of M_3. If not, M_3 must admit at least a $U(1) \times U(1)$ isometry and therefore has a *toric* almost contact structure.

We may then introduce two angular coordinates φ_1, φ_2 with period 2π, such that

$$K^{(3)} \;=\; b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} \ . \tag{1.22}$$

By the localization argument, the authors of [33] reduced the full partition function to a finite-dimensional integral. This reads

$$Z \;=\; \int \mathrm{d}\sigma_0 \; \mathrm{e}^{\frac{\mathrm{i}\pi k}{b_1 b_2}\mathrm{Tr}\sigma_0^2} \prod_{\alpha\in\Delta_+} 4\sinh\frac{\pi\sigma_0\alpha}{b_1}\sinh\frac{\pi\sigma_0\alpha}{b_2}\prod_{\rho} s_\beta\left(\frac{\mathrm{i}Q}{2}(1-r) - \frac{\rho(\sigma_0)}{\sqrt{b_1b_2}}\right) \ , \tag{1.23}$$

where the integral is over the Cartan of the gauge group, k denotes the Chern-Simons level, the first product is over positive roots $\alpha \in \Delta_+$ of the gauge group, and the second product is over weights ρ in the weight space decomposition for a chiral matter field with R-charge r in an arbitrary representation of the gauge group. Also, $\beta \equiv \sqrt{\frac{b_1}{b_2}}$ and $Q \equiv \beta + \frac{1}{\beta}$ and $s_\beta(z)$ is the double sine function defined by

$$s_\beta(z) \;\equiv\; \prod_{m,n\geq 0}\frac{\beta m + \beta^{-1}n + \frac{Q}{2} - \mathrm{i}z}{\beta m + \beta^{-1}n + \frac{Q}{2} + \mathrm{i}z} \ . \tag{1.24}$$

Notice in (1.23) that a factor of $1/\sqrt{b_1b_2}$ may be absorbed into σ_0, which is integrated over. Hence, the partition function depends on the background geometry only through one parameter, b_1/b_2.

In the context of the gauge/gravity duality, we are particularly interested in the free energy $\mathcal{F} = -\log Z$, which in the large N limit reads,

$$\lim_{N\to\infty}\mathcal{F}_\beta \;=\; \frac{(|b_1|+|b_2|)^2}{4|b_1b_2|}\mathcal{F}_{\beta=1} \ , \tag{1.25}$$

where $\mathcal{F}_{\beta=1}$ is the large N limit of the free energy on the round three-sphere scaling as $N^{3/2}$ [45].

$\mathcal{N}=1$ theories on $S^1\times S^3$

As mentioned, the recent interest in localization began with the computation of the partition function of $\mathcal{N}=2$ SYM on the four-sphere [36]. Localization has been used on other topologies in four-dimensions, such as $S^2\times S^2$ [53] and $T^2\times S^2$ [54].

In much of this thesis, we will focus on $S^1\times S^3$. In ref. [26], the authors computed the full partition function for $\mathcal{N}=1$ theories, consisting of vector multiplets and chiral multiplets with an R-symmetry, on a background with $S^1\times S^3$ topology. The theories considered in [26] are precisely those discussed in section 1.2.1 above. Analogously to the three-dimensional case above, the S^3 is assumed to have a $U(1)\times U(1)$ isometry. Introducing standard 2π-period toric coordinates φ_1, φ_2, the supersymmetric Killing vector (1.4) can be parametrized by two coefficients, $\mathfrak{b}_1, \mathfrak{b}_2$

$$K = \frac{1}{2}\left[\mathfrak{b}_1\partial_{\varphi_1} + \mathfrak{b}_2\partial_{\varphi_2} - \mathrm{i}\partial_\tau\right] , \tag{1.26}$$

with τ the coordinate on S^1. The partition function was found to be of the form,

$$Z[\mathfrak{b}_1, \mathfrak{b}_2] = \mathrm{e}^{-\mathcal{F}(\mathfrak{b}_1,\mathfrak{b}_2)}\mathcal{I}(\mathfrak{b}_1, \mathfrak{b}_2) , \tag{1.27}$$

where $\mathcal{I}(\mathfrak{b}_1, \mathfrak{b}_2)$ is the so-called supersymmetric index [55–58], and

$$\mathcal{F}(\mathfrak{b}_1, \mathfrak{b}_2) = \frac{4\pi}{3}\left(|\mathfrak{b}_1| + |\mathfrak{b}_2| - \frac{|\mathfrak{b}_1| + |\mathfrak{b}_2|}{|\mathfrak{b}_1\mathfrak{b}_2|}\right)(\mathbf{a} - \mathbf{c}) + \frac{4\pi}{27}\frac{(|\mathfrak{b}_1| + |\mathfrak{b}_2|)^3}{|\mathfrak{b}_1\mathfrak{b}_2|}(3\mathbf{c} - 2\mathbf{a}) , \tag{1.28}$$

with $\mathbf{a}$ and $\mathbf{c}$ the trace anomaly coefficients. These are given by

$$\mathbf{a} = \frac{3}{32}\left(3\mathrm{tr}\mathbf{R}^3 - \mathrm{tr}\mathbf{R}\right) , \qquad \mathbf{c} = \frac{1}{32}\left(9\mathrm{tr}\mathbf{R}^3 - 5\mathrm{tr}\mathbf{R}\right) , \tag{1.29}$$

where $\mathbf{R}$ denotes the R-symmetry charge, and "tr" runs over the fermionic fields in the multiplets, so that for N_v vector multiplets and N_χ chiral multiplets,

$$\mathrm{tr}\mathbf{R}^n = N_v + \sum_{I=1}^{N_\chi}(r_I - 1)^n . \tag{1.30}$$

We shall return to the partition function (1.27) shortly.

1.4 Gauge/gravity duality with curved boundaries

The first example of a dual gravity description of a gauge theory on a curved background appeared in [59, 60]. In particular, in [60] the authors constructed a supergravity solution dual to four-dimensional $\mathcal{N} = 1$ pure super-Yang-Mills theory, living in the unwrapped dimensions of a D5-brane wrapping a two-cycle inside a Calabi-Yau three-fold[3].

In ref. [45] the free energy of ABJM theory on the round three-sphere was computed. The authors further compared the large-N limit of the free energy to the holographically renormalized on-shell action of gravity on Euclidean AdS_4, which reads

$$S^{\mathrm{ren}}_{\mathrm{sugra}} = \frac{\pi}{2G_4} , \tag{1.31}$$

where G_4 is Newton's constant in four dimensions. This gave a precise match.

As new exact results for supersymmetric field theories on non-trivially curved backgrounds were obtained using localization, this prompted the study of the

[3] In this, and similar constructions, the unwrapped directions of the branes, on which the field theories live, are not curved.

gauge/gravity duality in such settings. When the field theory is defined on a conformally flat Riemannian manifold, the gravity dual must be asymptotically Euclidean anti-de Sitter (Euclidean AAdS). More generally, the gravity dual of field theories on non-conformally flat backgrounds will be just asymptotically *locally* Euclidean anti-de Sitter (Euclidean AlAdS).

This programme was initiated in [61] where a supersymmetric Euclidean AlAdS solution of four-dimensional minimal gauged supergravity was proposed as the dual to three-dimensional Chern-Simons theories defined on a one-parameter squashed three-sphere. The localized partition function of such theories had previously been computed in [49]. Generalizations have been discussed in [62, 63], and a two-parameter squashing was presented in [64]. In all these cases, the gravity duals are supersymmetric solutions of four-dimensional $\mathcal{N} = 2$ minimal gauged supergravity in Euclidean signature. They are comprised of a negatively curved Einstein anti-self-dual metric on the four-ball [4], with a specific choice of anti-self-dual gauge field. The concrete check was the comparison of the holographically renormalized on-shell action with the free energy of the field theory.

Further examples of four-dimensional gravity solutions with curved boundary, where localization was utilized in the dual field theory, have been discussed in [65,66]. In this case, the exactly calculable quantity on both sides of the duality is the so-called supersymmetric Rényi entropy [67], which is a simple modification of the partition function on the ellipsoid [49] (see also [61]).

The most general example in four bulk dimensions was given in [1]. This reference presented the gravity duals to $\mathcal{N} = 2$ Chern-Simons theories on an arbitrary toric metric on the three-sphere. As described in the previous section, the localized partition function of such theories was computed in [33], leading to the free energy (1.25). The gravity solution in [1] is again a supersymmetric AlAdS solution of $\mathcal{N} = 2$ minimal gauged supergravity, and it encompasses all known such solutions. It has anti-self-dual Weyl tensor and is equipped with a gauge field with anti-self-dual field strength. From the Killing spinors, one can construct as a bilinear a Killing vector $\mathcal{K}$. In terms of the toric coordinate, φ_1 and φ_2, this can be parametrized as

$$\mathcal{K} \; = \; b_1\partial\varphi_1 + b_2\partial\varphi_2 \; . \tag{1.32}$$

On the conformal boundary, $\mathcal{K}$ becomes the Killing vector (1.22). The holographically renormalized on-shell action was shown to be

$$S^{\text{ren}}_{\text{sugra}} \; = \; \frac{\pi}{2G_4} \cdot \frac{(|b_1| + |b_2|)^2}{4|b_1 b_2|} \; , \tag{1.33}$$

[4] References [62, 63] also discuss several solutions with topology different from the four-ball; however the precise field theory duals of these remain unknown. In chapter 2, we will only be concerned with gravity solutions with topology of the four-ball.

with G_4 Newton's constant, precisely matching the expectation from localization (1.25). This constitutes a quite general check of the gauge/gravity duality. The details of this general gravity solution will be the topic of chapter 2.

In five bulk dimensions, gravity duals of $\mathcal{N} = 1$ SCFTs on $S^1 \times S^3$ with a one-parameter squashing of the S^3 have been constructed in [68], and the holographically renormalized on-shell action computed. As discussed in the next section, it remains an open problem to holographically match the *supersymmetric Casimir energy* of such field theories, a problem recently addressed in [4].

There are also results in six bulk dimensions. In [69], the gravity duals of supersymmetric gauge theories on a squashed five-sphere have been constructed in Romans $F(4)$ gauged supergravity. The holographic free energy and BPS Wilson loops were successfully matched to the five-dimensional localization computations.

1.5 Supersymmetric Casimir energy

In this section, we turn to discuss properties of the Casimir energy of superconformal field theories, *i.e.* the energy of the vacuum. The Casimir energy can be expressed in terms of the trace anomaly coefficients, which appear in the trace of the energy-momentum tensor and encode universal properties of CFTs. In two dimensions, the trace anomaly is proportional to the central charge $\mathbf{c}$,

$$\langle T_\mu{}^\mu \rangle \;=\; -\frac{\mathbf{c}}{24\pi} R \,, \tag{1.34}$$

where R is the Ricci scalar of the background. The central charge $\mathbf{c}$ characterizes two-dimensional CFTs, and constrains the renormalization group flows between them [70]. In four-dimensional CFTs, there are two trace anomaly coefficients, $\mathbf{a}$ and $\mathbf{c}$, and we defined them already in (1.29).

Given a CFT on $\mathbb{R}^d$, we can use a Weyl transformation to place the theory on $\mathbb{R} \times S^{d-1}$, where the sphere is round. Denoting the non-compact coordinate by τ, we define the Casimir energy E_0 as

$$E_0 \;=\; \int_{S^{d-1}} \mathrm{d}^{d-1} x \sqrt{g} \langle T_{\tau\tau} \rangle \,, \tag{1.35}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, and the expectation value is taken in the ground state of the theory. The evaluation of the Casimir energy leads to infinite sums or products which must be regularized, for example using zeta functions such as the Riemann, Hurwitz, or Barnes zeta functions. However, the result will in general depend on the chosen regularization scheme and is therefore ambiguous.

Another way of regarding this regularization is by adding counterterms to the action. These may cancel the divergences, however, dimensionless counterterms will affect the finite part of the computation. In two dimensions there is only one dimensionless counterterm, namely

$$\int \mathrm{d}^2x\sqrt{g}R \ , \tag{1.36}$$

where R is the Ricci scalar of the background. This vanishes on the cylinder, $\mathbb{R}\times S^1$, and therefore the Casimir energy is well-defined and scheme-independent. It is well known that the Casimir energy for a CFT on $\mathbb{R}\times S^1$ is [71]

$$E_0^{\mathbb{R}\times S^1} \ = \ -\frac{\mathbf{c}}{12r_1} \ , \tag{1.37}$$

where r_1 is the radius of the S^1.

In four dimensions there are several dimensionless counterterms. A basis for these is given by the square of the Ricci scalar R^2, the square of the Weyl tensor $W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$, the Euler density $\mathscr{E} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$, and the Pontryagin density $\mathscr{P} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\kappa}R_{\lambda\kappa\rho\sigma}R^{\mu\nu\rho\sigma}$. On the background $\mathbb{R}\times S^3$ the only non-vanishing of these is R^2. Hence, we may add to the action a term

$$\delta S \ = \ -\frac{\mathbf{b}}{12(4\pi)^2}\int \mathrm{d}^4x\sqrt{g}R^2 \ , \tag{1.38}$$

with an arbitrary coefficient $\mathbf{b}$. This coefficient shows up in the trace of the energy-momentum tensor,[5]

$$\langle T_\mu{}^\mu\rangle \ = \ \frac{1}{(4\pi)^2}\left(\mathbf{a}\mathscr{E} - \mathbf{c}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \mathbf{b}\Box R\right) \ , \tag{1.39}$$

along with the trace anomaly coefficients $\mathbf{a}$ and $\mathbf{c}$, and shifts the Casimir energy of the CFT on $\mathbb{R}\times S^3$,

$$E_0^{\mathbb{R}\times S^3} \ = \ \frac{3}{4r_3}\left(\mathbf{a} - \frac{\mathbf{b}}{2}\right) \ . \tag{1.40}$$

A self-contained derivation of this result can be found in the appendix of [3].

This discussion was on general CFTs, not necessarily supersymmetric ones. However, for four-dimensional supersymmetric CFTs, a quantity can be defined dubbed the *supersymmetric Casimir energy* E_{susy}, which is in fact free of ambiguities [3].

From the path integral on a manifold of the form $S^1\times M_3$ with M_3 some three-manifold, the Casimir energy E_0 may be defined in the limit where the radius β of the S^1 is taken to infinity,

$$E_0 \ = \ -\lim_{\beta\to\infty}\frac{\mathrm{d}}{\mathrm{d}\beta}\log Z[\beta, M_3] \ . \tag{1.41}$$

In ref. [26], the authors computed the full partition function for $\mathcal{N} = 1$ theo-

[5] Although the right hand side of (1.39) vanishes for the conformally flat metric on $\mathbb{R}\times S^3$.

ries with an R-symmetry, consisting of vector multiplets and chiral multiplets on a background of $S^1 \times S^3$ topology. This led to the result (1.27) above. Here we are particularly interested in the round sphere $M_3 = S^3$ where we set $|\mathfrak{b}_1| = |\mathfrak{b}_2| = \frac{\beta}{2\pi}$. In this case, from equation (1.28), $\mathcal{F}$ simplifies to

$$\mathcal{F}(\beta) \;=\; \frac{4}{27}\beta\left(\mathbf{a}+3\mathbf{c}\right) - \frac{4}{3\beta}(\mathbf{a}-\mathbf{c}) \;, \tag{1.42}$$

Inserting the supersymmetric partition function (1.27) into equation (1.41), it can be shown that the supersymmetric index $\mathcal{I}(\beta)$ does not contribute when taking β to infinity. The only contribution comes from (1.42) and one finds the Casimir energy,

$$E_{\text{susy}} \;=\; \frac{4}{27}\left(\mathbf{a}+3\mathbf{c}\right) \;, \tag{1.43}$$

which is the supersymmetric Casimir energy.

In chapter 3, we focus on the supersymmetric Casimir energy following [2,3]. We consider the theories of section 1.2.1 to quadratic order in the fields, as we are concerned with the vacuum energy. From both a Euclidean path integral approach on $S^1 \times S^3$, as well as canonical quantization in Lorentzian signature on $\mathbb{R}\times S^3$, adopting a specific choice of regularization, we recover (1.43). We then argue that in fact the supersymmetric Casimir energy is free of ambiguities, provided the regularization scheme preserves supersymmetry.

1.5.1 Holography and the supersymmetric Casimir energy

According to the gauge/gravity duality, $\mathcal{N}=1$ superconformal field theories on $\mathbb{R}\times S^3$ have a dual description in terms of supersymmetric solutions of five-dimensional supergravity. However, the gravity dual reproducing the supersymmetric Casimir energy (1.43) remains to be identified.

The appropriate gravity solution must admit a gauge field coupling to the R-symmetry of the boundary field theory. For the conformally flat case of $\mathbb{R}\times S^3$, an obvious candidate for the gravity dual is pure AdS$_5$. Indeed, the boundary is $\mathbb{R}\times S^3$ and a constant gauge field $A = c\,\mathrm{d}t$ can be turned on. However, as we will further comment on in chapter 4, the field theory requires an electric charge in the bulk, in turn requiring a non-trivial gauge field so that the solution is only asymptotically AdS.

The conditions for obtaining supersymmetric solutions to minimal gauged supergravity in five dimensions were presented about a decade ago in [32]. By assuming the existence of a Killing spinor, the authors constructed bilinears of this spinor, leading to constraints on the metric and graviphoton. Solutions fall in two distinct classes depending on whether the supersymmetric Killing vector is timelike or null. In chapter 4 we will review these constraints in the timelike case. This formalism was used to construct the first example of an AAdS_5 black hole free of closed timelike curves [72]. Other AAdS_5 solutions were obtained by different methods in [73–76], with the solution of [76] being the most general in that it encompasses the others as special cases. The solution of [76] also contains the most general AAdS_5 black hole known within minimal gauged supergravity. It was shown in [77] that in the supersymmetric limit this black hole takes the form of the timelike class of [32]. The formalism of [32] also led to the construction of AlAdS_5 solutions in the timelike case [68, 78, 79] and the null case [80] (the latter based on [81]). Solutions of five-dimensional minimal gauged supergravity with an $SU(2) \times U(1) \times U(1)$ isometry where studied in [68].

We report in chapter 4 on an attempt to match $\mathcal{N} = 1$ superconformal field theories on $\mathbb{R} \times S^3$ with a supersymmetric AAdS_5 solution. This smooth supersymmetric solution known as a *topological soliton* was first found in [76]. This is based on ref. [4].

Chapter 2

Gravity duals of supersymmetric gauge theories on three-manifolds

In this chapter we construct the gravity duals of supersymmetric gauge theories on three-manifolds. In section 2.1 we present a supersymmetric solution of four-dimensional minimal gauged supergravity in Euclidean signature, comprising a metric with anti-self-dual Weyl tensor and a graviphoton with anti-self-dual field strength, and we find explicitly the spinor ϵ that solves the Killing spinor equation. In section 2.2 we discuss regularity of the solution, assuming topology of the four-ball, and show that the conformal boundary is of the form discussed in section 1.2.2, for which the localized partition function was computed in [33]. Assuming at least a $U(1) \times U(1)$ isometry, we compute in section 2.3 the holographically renormalized on-shell action. We arrive at the expression advertised earlier in equation (1.33), matching the field theory result (1.25). We then discuss in section 2.4 previously known explicit examples, which are obtained as special cases of our solution. We end this chapter with a discussion of possible generalizations in section 2.5. This chapter is based on [1].

2.1 Local geometry of self-dual solutions

The action of the bosonic sector of four-dimensional $\mathcal{N} = 2$ gauged supergravity [82] in Euclidean signature is

$$S_{\text{sugra}} \;=\; -\frac{1}{16\pi G_4}\int \left(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}\right)\sqrt{g}\,\mathrm{d}^4x \;, \qquad (2.1)$$

where R denotes the Ricci scalar of the four-dimensional metric $g_{\mu\nu}$. Throughout this chapter, we normalize the cosmological constant to $\Lambda = -3$. The graviphoton is an Abelian gauge field A with field strength $F = \mathrm{d}A$. The equations of motion derived from (2.1) are

$$
\begin{aligned}
R_{\mu\nu} + 3g_{\mu\nu} &= 2\left(F_\mu{}^\rho F_{\nu\rho} - \tfrac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}\right) \ , \\
\mathrm{d} *_4 F &= 0 \ .
\end{aligned}
\tag{2.2}
$$

This is simply Einstein-Maxwell theory with a cosmological constant. When F is anti-self-dual, $*_4F = -F$, the right hand side of the Einstein equation in (2.2) is easily shown to vanish, so that the metric $g_{\mu\nu}$ is necessarily Einstein.

A solution is supersymmetric provided it admits a (not identically zero) Dirac spinor ϵ satisfying the Killing spinor equation

$$
\left(\nabla_\mu - \mathrm{i}A_\mu + \frac{1}{2}\Gamma_\mu + \frac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right)\epsilon \ = \ 0 \ . \tag{2.3}
$$

This takes the same form as in Lorentzian signature, except that here the gamma matrices generate the Clifford algebra Cliff$(4,0)$ in an orthonormal frame, so $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$. Notice that we may define the charge conjugate of the spinor ϵ as $\epsilon^c \equiv B\epsilon^*$, where B is the charge conjugation matrix satisfying $B^{-1}\Gamma_\mu B = \Gamma_\mu^*$, $BB^* = -1$ and may be chosen to be antisymmetric $B^T = -B$ [61]. Then provided the gauge field A is real (as we will assume here) ϵ^c satisfies (2.3) with $A \to -A$.

In [83, 84] the authors studied the local geometry of Euclidean supersymmetric solutions to the above theory for which F is anti-self-dual. It follows that the metric $g_{\mu\nu}$ has anti-self-dual Weyl tensor W,

$$
W_{\mu\nu\rho\sigma} \ = \ -\frac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\kappa}W_{\rho\sigma\lambda\kappa} \ . \tag{2.4}
$$

Adopting a standard abuse of terminology we shall refer to such solutions as "self-dual". Supersymmetry also equips this background geometry with a Killing vector field $\mathcal{K}$ constructed as a bilinear of the Killing spinor. Self-dual Einstein metrics with a Killing vector have a rich geometric structure that has been well-studied (see for example [85]) and shown to be related by a Weyl rescaling to a (local) Kähler metric with zero Ricci scalar. The metric is described by a function solving a single PDE, known as the Toda equation. This function also specifies uniquely the gauge field A. In fact we will show that $F = \mathrm{d}A$ is $\frac{1}{2}$ times the Ricci-form of the conformally related Kähler metric, so that A is the natural connection on $\mathcal{C}^{-1/2}$, where $\mathcal{C}$ denotes the canonical bundle of the Kähler manifold. Moreover, we will reverse the direction of implication in [83, 84] and show that any self-dual Einstein metric with a choice of Killing vector field admits (locally) a solution to the Killing spinor equation (2.3). This may be constructed from the canonically defined spin^c spinor that exists on any Kähler manifold.

2.1.1 Local form of the solution

In this section we briefly review the local geometry determined in [83, 84]. The existence of a non-trivial solution to the Killing spinor equation (2.3), together with the ansatz that F is anti-self-dual and real, implies that the metric $g_{\mu\nu}$ is Einstein with anti-self-dual Weyl tensor. There is then a canonically defined local coordinate system in which the metric takes the form

$$\mathrm{d}s^2_{\mathrm{SDE}} = \frac{1}{y^2}\left[B^{-1}(\mathrm{d}\psi+\phi)^2 + B\left(\mathrm{d}y^2 + 4\mathrm{e}^w \mathrm{d}z\mathrm{d}\bar{z}\right)\right] , \tag{2.5}$$

where

$$B = 1 - \frac{1}{2}y\partial_y w , \tag{2.6}$$

$$\mathrm{d}\phi = \mathrm{i}\partial_z B \mathrm{d}y\wedge\mathrm{d}z - \mathrm{i}\partial_{\bar{z}}B\mathrm{d}y\wedge\mathrm{d}\bar{z} + 2\mathrm{i}\partial_y(B\mathrm{e}^w)\mathrm{d}z\wedge\mathrm{d}\bar{z} , \tag{2.7}$$

and $w = w(y, z, \bar{z})$ satisfies the Toda equation

$$\partial_z\partial_{\bar{z}}w + \partial_y^2\mathrm{e}^w = 0 . \tag{2.8}$$

Notice that the function w determines entirely the metric. The two-form $\mathrm{d}\phi$ is easily verified to be closed provided the Toda equation (2.8) is satisfied, implying the existence of a local one-form ϕ.

The vector $\mathcal{K} = \partial_\psi$ is a Killing vector field, and arises canonically from supersymmetry as a bilinear $\mathcal{K}^\mu \equiv \mathrm{i}\epsilon^\dagger\Gamma^\mu\Gamma_5\epsilon$, where ϵ is the Killing spinor solving (2.3) and $\Gamma_5 \equiv \Gamma_{0123}$. Notice that the corresponding bilinear in the charge conjugate spinor ϵ^c is $\mathrm{i}(\epsilon^c)^\dagger\Gamma^\mu\Gamma_5\epsilon^c = -\mathcal{K}^\mu$. Thus as in the discussion after equation (2.3) we may change variables to $\tilde{\epsilon} = \epsilon^c$, $\tilde{A} = -A$. In the tilded variables the equations of motion (2.2) and Killing spinor equation (2.3) are identical to the untilded equations, but now $\tilde{A} = -A$ and $\tilde{\mathcal{K}} = -\mathcal{K}$. Thus the sign of the instanton is correlated with a choice of sign for the supersymmetric Killing vector, with charge conjugation of the spinor changing the signs of both A and $\mathcal{K}$.

As we shall see in the next section, the coordinate y determines the conformal factor for the conformally related Kähler metric, and is also the Hamiltonian function for the vector field $\mathcal{K} = \partial_\psi$ with respect to the associated symplectic form. The graviphoton field is given by

$$A = -\frac{1}{4}B^{-1}\partial_y w(\mathrm{d}\psi+\phi) + \frac{\mathrm{i}}{4}\partial_z w\mathrm{d}z - \frac{\mathrm{i}}{4}\partial_{\bar{z}}w\mathrm{d}\bar{z} . \tag{2.9}$$

We are of course free to make gauge transformations of A, and we stress that (2.9) is in general valid only locally.

Having summarized the results of [83, 84], in the next two sections we study this local geometry further. In particular we show that any self-dual Einstein metric with Killing vector $\mathcal{K} \equiv \partial_\psi$, which then takes the form (2.5), admits a Killing spinor ϵ solving (2.3), where A is given by (2.9).

2.1.2 Conformal Kähler metric

As already mentioned, every self-dual Einstein four-metric with a Killing vector is conformally related to a scalar-flat Kähler metric. This is given by

$$\begin{aligned} \mathrm{d}s^2_{\text{Kähler}} &\equiv \mathrm{d}\hat{s}^2 = y^2 \mathrm{d}s^2_{\text{SDE}} \\ &= B^{-1}(\mathrm{d}\psi + \phi)^2 + B\left(\mathrm{d}y^2 + 4\mathrm{e}^w \mathrm{d}z\mathrm{d}\bar{z}\right) . \end{aligned} \tag{2.10}$$

Introducing an associated local orthonormal frame of one-forms,

$$\hat{e}^0 = B^{1/2}\mathrm{d}y , \quad \hat{e}^1 = B^{-1/2}(\mathrm{d}\psi + \phi) , \quad \hat{e}^2 + \mathrm{i}\hat{e}^3 = 2(B\mathrm{e}^w)^{1/2}\mathrm{d}z , \tag{2.11}$$

the Kähler form is

$$\omega = \hat{e}^{01} + \hat{e}^{23} , \tag{2.12}$$

where we have denoted $\hat{e}^0 \wedge \hat{e}^1 = \hat{e}^{01}$, *etc.* That (2.12) is indeed closed follows immediately from the expression for $\mathrm{d}\phi$ in (2.7). The Kähler form is self-dual with respect to the natural orientation on a Kähler manifold, namely $\hat{e}^{0123}$ above, and it is with respect to this orientation that the curvature F and Weyl tensor are anti-self-dual. We denote the corresponding orthonormal frame for the self-dual Einstein metric (2.5) as $e^a = y^{-1}\hat{e}^a$, $a = 0, 1, 2, 3$.

Next we introduce the Hodge type $(2,0)$-form

$$\Omega \equiv (\hat{e}^0 + \mathrm{i}\hat{e}^1) \wedge (\hat{e}^2 + \mathrm{i}\hat{e}^3) , \tag{2.13}$$

and recall that the metric (2.10) is Kähler if and only if

$$\mathrm{d}\Omega = \mathrm{i}\mathcal{P} \wedge \Omega , \tag{2.14}$$

where $\mathcal{P}$ is then the Ricci one-form, with Ricci two-form $\mathcal{R} = \mathrm{d}\mathcal{P}$. Recall that $\mathcal{R}_{\mu\nu} = \frac{1}{2}\hat{R}_{\mu\nu\rho\sigma}\omega^{\rho\sigma}$ where $\hat{R}_{\mu\nu\rho\sigma}$ denotes the Riemann tensor for the Kähler metric. It is straightforward to compute $\mathrm{d}\Omega$ for the metric (2.10), and one finds that

$$\mathcal{P} = 2A , \tag{2.15}$$

where A is given by (2.9). Thus the gauge field is the natural connection on $\mathcal{C}^{-1/2}$, where $\mathcal{C}$ denotes the canonical line bundle for the Kähler metric. The curvature is correspondingly $F = \mathrm{d}A = \frac{1}{2}\mathcal{R}$. A computation gives

$$-2\mathcal{R}\wedge\omega \;=\; \frac{1}{B\mathrm{e}^w}\left[\partial_z\partial_{\bar{z}}w + \partial_y^2\mathrm{e}^w\right]\hat{e}^{0123}\;, \tag{2.16}$$

so that the Kähler metric is indeed scalar flat if the Toda equation holds. Since the Ricci two-form is Hodge type $(1,1)$ and the metric is scalar flat, it follows immediately that $F = \frac{1}{2}\mathcal{R}$ is anti-self-dual. This is because the anti-self-dual two-forms on a Kähler four-manifold are precisely the primitive $(1,1)$-forms (*i.e.* having zero wedge product with ω, as in (2.16)), so $\Lambda^2_- \cong \Lambda_0^{(1,1)}$. An explicit computation shows that with respect to the frame (2.11)

$$\begin{aligned}F \;=\; & -\frac{1}{4}\partial_y\left[B^{-1}\partial_y w\right]\left(\hat{e}^{01}-\hat{e}^{23}\right)+\frac{1}{8\mathrm{e}^{w/2}}\Big[\mathrm{i}(\partial_z-\partial_{\bar{z}})[B^{-1}\partial_y w]\left(\hat{e}^{02}+\hat{e}^{13}\right)\\ & -(\partial_z+\partial_{\bar{z}})[B^{-1}\partial_y w]\left(\hat{e}^{03}-\hat{e}^{12}\right)\Big]\;,\end{aligned} \tag{2.17}$$

which is then manifestly anti-self-dual. One can also derive the formula

$$F \;=\; -\left(\tfrac{1}{2}y\mathrm{d}\mathcal{K} + y^2\mathcal{K}\wedge J\mathcal{K}\right)^-\;, \tag{2.18}$$

where here we mean by $\mathcal{K} = g_{\mu\nu}\mathcal{K}^\nu\mathrm{d}x^\mu$ the one-form dual to the Killing vector $\mathcal{K}^\mu$ (in the self-dual Einstein metric), and J is the complex structure tensor for the Kähler metric (2.10). A further short computation leads to

$$F \;=\; \left(\frac{1}{y}\mathrm{i}\partial\bar{\partial}y\right)^- \;=\; \frac{1}{y}\mathrm{i}\partial\bar{\partial}y + \frac{1}{4y}(\hat{\Delta}y)\omega\;, \tag{2.19}$$

where $\bar{\partial}$ denotes the standard operator on a Kähler manifold, the superscript "$-$" in (2.19) denotes anti-self-dual part, and $\hat{\Delta}$ denotes the scalar Laplacian for the Kähler metric.

Let us note that the Kähler form is explicitly

$$\omega \;=\; \mathrm{d}y\wedge(\mathrm{d}\psi+\phi) + 2\mathrm{i}B\mathrm{e}^w\mathrm{d}z\wedge\mathrm{d}\bar{z}\;. \tag{2.20}$$

Thus $\mathrm{d}y = -\partial_\psi \lrcorner\, \omega$, which identifies the coordinate y as the Hamiltonian function for the Killing vector $\mathcal{K} = \partial_\psi$. Of course, y^2 is also the conformal factor relating the self-dual Einstein metric to the Kähler metric in (2.10).

2.1.3 Killing spinor: sufficiency

In this section we show that a self-dual Einstein metric with Killing vector $\mathcal{K} = \partial_\psi$, which necessarily takes the form (2.5), admits a solution to the Killing spinor equation (2.3) with gauge field given by (2.9). The key to this construction is to begin with the canonically defined spin^c spinor that exists on any Kähler manifold.

The positive chirality spin bundle on a Kähler four-manifold takes the form $\mathcal{S}_+ \cong \mathcal{C}^{1/2} \oplus \mathcal{C}^{-1/2}$, where $\mathcal{C}$ denotes the canonical bundle. The spin bundle then exists globally only if the latter admits a square root, but the spin^c bundle $\mathcal{S}_+ \otimes \mathcal{C}^{-1/2} \cong 1 \oplus \mathcal{C}^{-1}$ always exists globally. In particular the first factor in $\mathcal{S}_+ \otimes \mathcal{C}^{-1/2} \cong 1 \oplus \mathcal{C}^{-1}$ is a trivial complex line bundle, whose sections may be identified with complex-valued functions, and there is always a section ζ satisfying the spin^c Killing spinor equation

$$\left(\hat{\nabla}_\mu - \tfrac{\mathrm{i}}{2}\mathcal{P}_\mu\right)\zeta \;=\; 0 \; . \tag{2.21}$$

Here the hat denotes that we will apply this to the conformal Kähler metric (2.10) in the case at hand, and $\mathcal{P}$ is the Ricci one-form potential we encountered above. The connection term in (2.21) precisely corresponds to twisting the spin bundle $\mathcal{S}_+$ by $\mathcal{C}^{-1/2}$. Using the result earlier that $\mathcal{P} = 2A$ the spin^c equation (2.21) may be rewritten as

$$\left(\hat{\nabla}_\mu - \mathrm{i}A_\mu\right)\zeta \;=\; 0 \; , \tag{2.22}$$

which may already be compared with the Killing spinor equation (2.3).

More concretely, the solution to (2.21), or equivalently (2.22), is simply given by a constant spinor ζ, so that $\partial_\mu \zeta = 0$. This equation makes sense globally as ζ may be identified with a complex-valued function. To see this it is useful to take the following projection conditions

$$\hat{\Gamma}_1\zeta \;=\; \mathrm{i}\hat{\Gamma}_0\zeta \; , \qquad \hat{\Gamma}_3\zeta \;=\; \mathrm{i}\hat{\Gamma}_2\zeta \; , \tag{2.23}$$

following *e.g.* reference [31]. Here $\hat{\Gamma}_a$, $a = 0,1,2,3$, denote the gamma matrices in the orthonormal frame (2.11).[1] The covariant derivative of ζ is then computed to be

$$\hat{\nabla}_\mu\zeta \;=\; \left(\partial_\mu + \frac{1}{4}\hat{\omega}_\mu{}^{\nu\rho}\hat{\Gamma}_{\nu\rho}\right)\zeta \;=\; \partial_\mu\zeta + \tfrac{\mathrm{i}}{2}\left(\hat{\omega}_\mu^{\;01} + \hat{\omega}_\mu^{\;23}\right)\zeta \;=\; \partial_\mu\zeta + \mathrm{i}A_\mu\zeta \; , \tag{2.24}$$

where $\hat{\omega}_\mu{}^{\nu\rho}$ is the spin connection of the conformal Kähler metric. We used here the expression (2.9) for A, as well as the explicit form of the spin connection given in the appendix of [1]. It follows that simply taking ζ to be constant, $\partial_\mu\zeta = 0$, solves (2.21). This is a general phenomenon on any Kähler manifold.

Using the canonical spinor ζ we may construct a spinor ϵ that is a solution to the Killing spinor equation (2.3). Specifically, we find

$$\epsilon \;=\; \frac{1}{\sqrt{2y}}\left(1 + B^{-1/2}\hat{\Gamma}_0\right)\zeta \; . \tag{2.25}$$

[1] Strictly speaking the hats are redundant, but we keep them as a reminder that in this section the orthonormal frame is for the Kähler metric.

To verify this one first notes that the spin connections of the Kähler metric and the self-dual Einstein metric are related by

$$\hat{\nabla}_\mu \zeta \;=\; \nabla_\mu \zeta + \frac{1}{2}\hat{\Gamma}_\mu{}^\nu(\partial_\nu \log y)\zeta \;, \tag{2.26}$$

where $\hat{\Gamma}_\mu = y\Gamma_\mu$ in a coordinate basis. The Killing spinor equation then takes the form

$$\left[\partial_\mu + \frac{1}{4}\hat{\omega}_\mu{}^{\nu\rho}\hat{\Gamma}_{\nu\rho} - \frac{1}{2}\hat{\Gamma}_\mu{}^\nu(\partial_\nu \log y) \;-\mathrm{i}A_\mu + \frac{1}{2y}\hat{\Gamma}_\mu + \frac{\mathrm{i}}{4}yF_{\nu\rho}\hat{\Gamma}^{\nu\rho}\hat{\Gamma}_\mu\right]\epsilon \;=\; 0\;. \tag{2.27}$$

To verify this is solved by (2.25) one simply substitutes (2.25) directly into the left-hand-side of (2.27). Using the explicit expressions for the spin connection, the gauge field, the field strength, as well as the projection conditions on the canonical spinor ζ and (2.21), one finds that (2.27) indeed holds.

From this analysis we can conclude that the self-dual Einstein metric (2.5) and the gauge field (2.9), which are solutions to Einstein-Maxwell theory in four dimensions, yield a Dirac spinor ϵ that is a solution to the Killing spinor equation (2.3). This implies that these self-dual Einstein backgrounds are always locally supersymmetric solutions of Euclidean $\mathcal{N}=2$ gauged supergravity. We turn to global issues in the next section.

2.2 Asymptotically locally AdS solutions

In this section and the next we will assume that we are given a complete (non-singular) self-dual Einstein metric with a Killing vector, which then necessarily takes the local form (2.5). Moreover, we shall assume this metric is asymptotically locally Euclidean AdS,[2] and in later subsections also that the four-manifold M_4 on which the metric is defined is topologically a ball. A two-parameter family of such self-dual solutions on the four-ball, generalizing all previously known solutions of this type, was constructed in [64]. In section 2.4 we shall review these solutions, and also introduce a number of further generalizations. In particular, the results of the current section allow us to deform the choice of Killing vector (which was essentially fixed in previous results), and we will also explain how to generalize to an *infinite-dimensional* family of solutions satisfying the above properties, starting with the local metrics in [86].

With the above assumptions in place, we begin in this section by showing that if the Killing vector $\mathcal{K} = \partial_\psi$ is nowhere zero in a neighbourhood of the conformal boundary three-manifold M_3 then it is a Reeb vector field for an almost contact structure on M_3. We then reproduce the same geometric structure on M_3 studied

[2]Since the metric has Euclidean signature one might more accurately describe this boundary condition as *asymptotically locally hyperbolic*, which is often used in the mathematics literature.

from a purely three-dimensional viewpoint in [24] and reviewed in section 1.2.2. In particular the asymptotic expansion of the Killing spinor ϵ leads to a Killing spinor equation of the form (1.11). This is important, as it shows that the dual field theory is defined on a supersymmetric background of the form studied in [24], for which the exact partition function of a general $\mathcal{N}=2$ supersymmetric gauge theory was computed in [33] using localization. Having studied the conformal boundary geometry, we then turn to the bulk in section 2.2.4. In particular we show that, with an appropriate restriction on the Killing vector $\mathcal{K}$, the conformal Kähler structure of section 2.1.2 is everywhere non-singular. This allows us to prove in turn that the instanton and Killing spinor defined by the Kähler structure are everywhere non-singular.

In particular this means that each of the self-dual Einstein metrics in section 2.4 leads to a one-parameter family (depending on the choice of Killing vector $\mathcal{K}$) of smooth supersymmetric solutions. In other words, if the self-dual Einstein metric depends on n parameters, the complete solution will depend on $n+1$ parameters. We emphasize that in the previously known solutions the only example of this phenomenon is the solution of [61]. There the Einstein metric was simply AdS_4, which doesn't have any parameters.

2.2.1 Conformal boundary at $y=0$

We are interested in self-dual Einstein metrics of the form (2.5) which are asymptotically locally Euclidean AdS (hyperbolic), in order to apply to the gauge/gravity duality. From the assumptions described above there is a single asymptotic region where the metric approaches $\frac{\mathrm{d}r^2}{r^2}+r^2\mathrm{d}s^2_{M_3}$ as $r\to\infty$, where M_3 is a smooth compact three-manifold. In fact the metrics (2.5) naturally have such a conformal boundary at $y=0$. More precisely, we impose boundary conditions such that $w(y,z,\bar{z})$ is analytic around $y=0$, so

$$w(y,z,\bar{z}) \;=\; w_{(0)}(z,\bar{z})+yw_{(1)}(z,\bar{z})+\tfrac{1}{2}y^2w_{(2)}(z,\bar{z})+\mathcal{O}(y^3)\ . \qquad (2.28)$$

It follows that

$$B(y,z,\bar{z}) \;=\; 1-\tfrac{1}{2}yw_{(1)}(z,\bar{z})-\tfrac{1}{2}y^2w_{(2)}(z,\bar{z})+\mathcal{O}(y^3)\ , \qquad (2.29)$$

and that the metric (2.5) is

$$\mathrm{d}s^2_{\mathrm{SDE}} \;=\; [1+\mathcal{O}(y)]\,\frac{\mathrm{d}y^2}{y^2}+\frac{1}{y^2}\left[(\mathrm{d}\psi+\phi_{(0)})^2+4\mathrm{e}^{w_{(0)}}\mathrm{d}z\mathrm{d}\bar{z}+\mathcal{O}(y)\right]\ . \qquad (2.30)$$

Here we have also expanded the one-form tangent to M_3

$$\phi(y,z,\bar{z})\,|_{M_3} \;=\; \phi_{(0)}(z,\bar{z})+y\phi_{(1)}(z,\bar{z})+\mathcal{O}(y^2). \qquad (2.31)$$

In fact by expanding (2.7) one can show that $\phi_{(1)} = 0$. Setting $r = 1/y$ this is to leading order

$$\mathrm{d}s^2_{\mathrm{SDE}} \simeq \frac{\mathrm{d}r^2}{r^2} + r^2 \left[(\mathrm{d}\psi + \phi_{(0)})^2 + 4\mathrm{e}^{w_{(0)}}\mathrm{d}z\mathrm{d}\bar{z}\right] , \tag{2.32}$$

as $r \to \infty$, so that the metric is indeed asymptotically locally Euclidean AdS around $y = 0$. Of course, as usual one is free to redefine the radial coordinate $r \to r\Theta(\psi, z, \bar{z})$, where Θ is any smooth, nowhere zero function on M_3, resulting in a conformal transformation of the boundary metric $\mathrm{d}s^2_{M_3} \to \Theta^2\mathrm{d}s^2_{M_3}$. However, in the present context notice that $r = 1/y$ is a *natural* choice of radial coordinate.

With the analytic boundary condition (2.28) for w it follows automatically that $\mathcal{K} = \partial_\psi$ is nowhere zero in a neighbourhood of the conformal boundary $y = 0$. As we shall see, this will reproduce the same structure on M_3 as [24], but we should stress that this is not the general situation. For example, one could take the standard hyperbolic metric for Euclidean AdS, conformally embedded as a unit ball in $\mathbb{R}^4$, and take $\mathcal{K}$ to be the Killing vector that rotates the first factor in $\mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{R}^4$. The ansatz (2.28) is thus certainly a restriction on the class of possible globally regular solutions, although all examples in section 2.4 have choices of Killing vector for which this expansion holds.

Returning to the case at hand, the conformal boundary is a compact three-manifold M_3 (by assumption), and from the above discussion a natural choice of representative for the metric is

$$\mathrm{d}s^2_{M_3} = (\mathrm{d}\psi + \phi_{(0)})^2 + 4\mathrm{e}^{w_{(0)}}\mathrm{d}z\mathrm{d}\bar{z} . \tag{2.33}$$

Notice that the form of the metric (2.33) is precisely of the form (1.13), as studied in [33]. As discussed in section 1.2.2, an important role is played by the one-form

$$\eta \equiv \mathrm{d}\psi + \phi_{(0)} , \tag{2.34}$$

which has exterior derivative

$$\mathrm{d}\eta = \mathrm{d}\phi_{(0)} = 2\mathrm{i}\partial_y(B\mathrm{e}^w)\,|_{y=0}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} = \mathrm{i}w_{(1)}\mathrm{e}^{w_{(0)}}\mathrm{d}z \wedge \mathrm{d}\bar{z} . \tag{2.35}$$

The form η is a global *almost contact one-form* on M_3, see equation (1.15).

The Killing vector $\mathcal{K} = \partial_\psi$ is the *Reeb vector* for the almost contact form η, as follows from the equations

$$\mathcal{K}\lrcorner\eta = 1 , \qquad \mathcal{K}\lrcorner\mathrm{d}\eta = 0 . \tag{2.36}$$

The orbits of $\mathcal{K}$ thus foliate M_3, and moreover this foliation is transversely holomorphic with local complex coordinate z. When the orbits of $\mathcal{K}$ all close it generates a

$U(1)$ symmetry of the boundary structure, and the orbit space $M_3/U(1)$ is in general a compact orbifold surface, on which z may be regarded as a local complex coordinate. These are generally called Seifert fibred three-manifolds in the literature. On the other hand, if $\mathcal{K}$ has at least one non-closed orbit then, since the isometry group of a compact manifold is compact, we deduce that M_3 admits at least a $U(1)\times U(1)$ symmetry, and the structure defined by η is a *toric* almost contact structure. In this case we may introduce standard 2π-period coordinates φ_1, φ_2 on the torus $U(1)\times U(1)$ and write

$$\mathcal{K} \;=\; \partial_\psi \;=\; b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} \ . \tag{2.37}$$

From (2.35) we deduce that the Taylor coefficient $w_{(1)}$ is a globally defined *basic* function on M_3 – that is, it is invariant under $\mathcal{K}=\partial_\psi$. Moreover, the almost contact form η is a *contact form* precisely when the function $w_{(1)}$ is everywhere positive. We shall see later that there are examples for which η is contact and not contact. On the other hand, the coefficient $w_{(0)}$ is in general only a locally defined function of $z,\bar{z}$, as one sees by noting that the transverse metric $g_T = \mathrm{e}^{w_{(0)}}\mathrm{d}z\mathrm{d}\bar{z}$ is a global two-tensor, but in general the complex coordinate z is defined only locally.[3] It will be useful in what follows to define a corresponding transverse volume form

$$\mathrm{vol}_T \;\equiv\; 2\mathrm{i}\mathrm{e}^{w_{(0)}}\mathrm{d}z\wedge\mathrm{d}\bar{z} \ . \tag{2.38}$$

Again, this is a global tensor on M_3, with

$$\mathrm{d}\eta \;=\; \mathrm{d}\phi_{(0)} \;=\; \frac{w_{(1)}}{2}\mathrm{vol}_T \ . \tag{2.39}$$

2.2.2 Boundary Killing spinor

In this section we show that the Killing spinor ϵ induces a Killing spinor χ on the conformal boundary M_3 that solves the Killing spinor equation (1.11).

For the self-dual Einstein metric (2.5) we take the orthonormal frame

$$e^0 \;=\; \frac{1}{y}B^{1/2}\mathrm{d}y \ , \quad e^1 \;=\; \frac{1}{y}B^{-1/2}(\mathrm{d}\psi+\phi) \ , \quad e^2+\mathrm{i}e^3 \;=\; \frac{2}{y}(B\mathrm{e}^w)^{1/2}\mathrm{d}z \ . \tag{2.40}$$

Correspondingly, we take the following frame for the metric (2.33) on the three-dimensional boundary,

$$e^1_{(3)} \;=\; \mathrm{d}\psi+\phi_{(0)} \ , \qquad e^2_{(3)}+\mathrm{i}e^3_{(3)} \;=\; 2\mathrm{e}^{w_{(0)}/2}\mathrm{d}z \ , \tag{2.41}$$

[3]For example, for Euclidean AdS_4 realized as a hyperbolic ball and with $\mathcal{K}=\partial_\psi$ generating the Hopf fibration of the boundary S^3 then g_T is the standard metric on the round two-sphere, implying that $w_{(0)}(z,\bar{z}) = -2\log(1+|z|^2)$ which blows up at $z=\infty$ (which is a smooth copy of $S^1\subset M_3\cong S^3$).

and will use indices $i,j,k=1,2,3$ for this orthonormal frame.

We next expand the four-dimensional Killing spinor equation (2.3) as a Taylor series in y. One starts by noting that $\Gamma^\mu = e^\mu{}_a\Gamma^a = \mathcal{O}(y)$. But as $\Gamma_\mu = e^a{}_\mu\Gamma_a = \mathcal{O}(1/y)$ and the field strength expands as $F = F_{(0)} + yF_{(1)} + \mathcal{O}(y^2)$ we see that

$$\frac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu \;=\; \mathcal{O}(y) \; . \tag{2.42}$$

The Killing spinor equation becomes

$$\left[\nabla^{(3)}_\mu - \mathrm{i}A_{(0)\mu} + \frac{1}{2y}\left(1+\frac{1}{4}yw_{(1)}\right)e^i_{(3)\mu}(\Gamma_i - \Gamma_{i0}) + \mathcal{O}(y)\right]\epsilon \;=\; 0 \; , \tag{2.43}$$

where $\mu = \psi, z, \bar{z}$, and where

$$A_{(0)} \;=\; -\frac{1}{4}w_{(1)}e^1_{(3)} + \frac{\mathrm{i}}{8}\mathrm{e}^{-w_{(0)}/2}(\partial_z - \partial_{\bar{z}})w_{(0)}e^2_{(3)} - \frac{1}{8}\mathrm{e}^{-w_{(0)}/2}(\partial_z + \partial_{\bar{z}})w_{(0)}e^3_{(3)} \; , \tag{2.44}$$

is the lowest order term in the expansion of A given by (2.9). We emphasize again that this expression for $A_{(0)}$ is in general only valid locally. The Killing spinor ϵ then expands as

$$\epsilon \;=\; \frac{1}{\sqrt{2y}}\left[1 + \Gamma_0 + \frac{1}{4}yw_{(1)}\Gamma_0 + \mathcal{O}(y^2)\right]\zeta_0 \; , \tag{2.45}$$

where ζ_0 is the lowest order (y-independent) part of the Kähler spinor ζ. Substituting this into (2.43) gives a leading order term that is identically zero. The subleading term then reads

$$\left[\left(\nabla^{(3)}_i - \mathrm{i}A_{(0)i}\right)(1+\Gamma_0) + \frac{1}{8}w_{(1)}(\Gamma_{i0} - \Gamma_i)\right]\zeta_0 \;=\; 0 \; . \tag{2.46}$$

The projections (2.23), in the current context, read

$$\Gamma_1\zeta_0 \;=\; \mathrm{i}\Gamma_0\zeta_0 \; , \qquad\qquad \Gamma_3\zeta_0 \;=\; \mathrm{i}\Gamma_2\zeta_0 \; . \tag{2.47}$$

We may choose the following representation of the gamma matrices:

$$\Gamma_i \;=\; \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix} \; , \qquad\qquad \Gamma_0 \;=\; \begin{pmatrix} 0 & \mathrm{i}\mathbb{1}_2 \\ -\mathrm{i}\mathbb{1}_2 & 0 \end{pmatrix} \; , \tag{2.48}$$

with γ_i the Pauli matrices. The projection conditions then force ζ_0 to take the form

$$\zeta_0 \;=\; \begin{pmatrix} \chi \\ 0 \end{pmatrix} \quad \text{where} \quad \chi \;=\; \begin{pmatrix} \chi_0 \\ \chi_0 \end{pmatrix} \; . \tag{2.49}$$

Here χ is a two-component spinor and χ_0 is simply a constant. The three-dimensional

Killing spinor equation then becomes

$$\left(\nabla_i^{(3)} - \mathrm{i}A_{(0)i} - \frac{\mathrm{i}}{8}w_{(1)}\gamma_i\right)\chi \;=\; 0 \;. \tag{2.50}$$

Clearly, this equation is of the form (1.11), with[4] $A^{(3)} = A_{(0)}$, $h = -\frac{1}{4}w_{(1)}$, and $V^{(3)} = 0$. It is indeed important that our Killing spinor equation reproduces equation (1.11), so that the conformal boundary admits the $\mathcal{N} = 1$ field theories considered in [24], for which the localized partition function was computed in [33].

As already mentioned below (2.32), supersymmetry singled out a natural representative of the conformal class of the boundary metric. However, one is free to change the radial coordinate as $r \to \Theta r$, resulting in a conformal transformation of the boundary metric $\mathrm{d}s^2_{M_3} \to \Theta^2 \mathrm{d}s^2_{M_3}$. This also shifts the fields $A^{(3)}$, $V^{(3)}$, and h appearing in the Killing spinor equation (2.50). For further details on this see appendix B of [1].

2.2.3 Non-singular gauge

In a neighbourhood of the conformal boundary, the Kähler metric is defined on $[0, \epsilon) \times M_3$, for some $\epsilon > 0$. This follows since via the conformal rescaling (2.10) the Kähler metric asymptotes to

$$\mathrm{d}s^2_{\text{Kähler}} \;\simeq\; \mathrm{d}y^2 + \mathrm{d}s^2_{M_3} \;, \tag{2.51}$$

near to the conformal boundary $y = 0$. In particular the Kähler structure is smooth and globally defined in a neighbourhood of this boundary. Recall also that the gauge field A is a connection on $\mathcal{C}^{-1/2}$. Since every orientable three-manifold is spin, the canonical bundle $\mathcal{C}$ admits a square root in this neighbourhood, and so A restricts to a *bona fide* connection one-form on M_3. The corresponding $U(1)$ principal bundle can certainly be non-trivial for generic topology of M_3. In this section we analyze the simpler case where $M_3 \cong S^3$. Here A necessarily restricts to a *global* one-form $A_{(0)}$ on the conformal boundary, but as we shall see, the explicit representative (2.44) is in a singular gauge. Correspondingly, since the boundary Killing spinor χ is a spin^c spinor, the solution (2.49) to (2.50) is similarly in a singular gauge. In this section we correct this by writing $A_{(0)}$ as a global one-form on $M_3 \cong S^3$.

The expression (2.44) for the restriction of A to the conformal boundary is of course only well-defined up to gauge transformations. We may rewrite the expression in (2.44) as

$$A^{\text{local}}_{(0)} \;=\; -\frac{1}{4}w_{(1)}(\mathrm{d}\psi + \phi_0) + \frac{\mathrm{i}}{4}\partial_z w_{(0)}\mathrm{d}z - \frac{\mathrm{i}}{4}\partial_{\bar{z}} w_{(0)}\mathrm{d}\bar{z} \;, \tag{2.52}$$

[4]Note that the superscript on $A^{(3)}$ is that of section 1.2.2 to remind that $A^{(3)}$ is a three-dimensional field, while the subscript on $A_{(0)}$ refers to the lowest order in the asymptotic expansion of the graviphoton (2.9). We hope this does not cause confusion.

adding the superscript label "local" to emphasize that in general this is only a local one-form. The first term is $-\frac{1}{4}w_{(1)}\eta$, which is always a global one-form on M_3, independently of the topology of M_3. However, the last two terms are not globally defined in general. We may remedy this in the case where $M_3 \cong S^3$ by making a gauge transformation, adding an appropriate multiple of $\mathrm{d}\psi$,

$$A_{(0)} \;=\; -\frac{1}{4}w_{(1)}\eta + \varrho\left[\mathrm{d}\psi + \frac{\mathrm{i}}{4\varrho}\partial_z w_{(0)}\mathrm{d}z - \frac{\mathrm{i}}{4\varrho}\partial_{\bar{z}} w_{(0)}\mathrm{d}\bar{z}\right] , \qquad (2.53)$$

with ϱ a constant. This is then a global one-form on $M_3 \cong S^3$ if and only if the curvature two-form of the connection in square brackets lies in the same basic cohomology class as $\mathrm{d}\eta = \mathrm{d}\phi_0$. Concretely, we write

$$\varrho\mathrm{d}\psi + \frac{\mathrm{i}}{4}\partial_z w_{(0)}\mathrm{d}z - \frac{\mathrm{i}}{4}\partial_{\bar{z}} w_{(0)}\mathrm{d}\bar{z} \;\equiv\; \varrho\mathrm{d}\psi + \Xi \;\equiv\; \varrho\eta + \alpha , \qquad (2.54)$$

and compute

$$\begin{aligned} \mathrm{d}\Xi \;&=\; -\frac{\mathrm{i}}{2}\partial_z\partial_{\bar{z}} w_{(0)}\mathrm{d}z\wedge\mathrm{d}\bar{z} \;=\; \left(w_{(1)}^2 + w_{(2)}\right)\mathrm{e}^{w_{(0)}}\frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}\bar{z} \\ &=\; \frac{1}{4}\left(w_{(1)}^2 + w_{(2)}\right)\mathrm{vol}_T , \end{aligned} \qquad (2.55)$$

where we used the Toda equation (2.8) and Taylor expanded. Since η is a global one-form on $M_3 \cong S^3$, it follows that (2.53) is a global one-form precisely if α defined via (2.54) is a global *basic* one-form, *i.e.* α is invariant under $\mathcal{L}_{\partial_\psi}$ and satisfies $\partial_\psi \lrcorner \alpha = 0$. In this case we have

$$\int_{M_3}\eta\wedge\frac{1}{\varrho}\mathrm{d}\Xi \;=\; \int_{M_3}\eta\wedge\mathrm{d}\eta , \qquad (2.56)$$

which may be interpreted as saying that $[\frac{1}{\varrho}\mathrm{d}\Xi] = [\mathrm{d}\eta] \in H^2_{\text{basic}}(M_3) \cong \mathbb{R}$ lie in the same basic cohomology class. Indeed, this is the case if and only if $\frac{1}{\varrho}\mathrm{d}\Xi$ and $\mathrm{d}\eta$ differ by the exterior derivative of a global basic one-form.

The integral on the right hand side of (2.56) is the *almost contact volume* of M_3:

$$\mathrm{Vol}_\eta \;\equiv\; \int_{M_3}\eta\wedge\mathrm{d}\eta \;=\; \int_{M_3}\frac{w_{(1)}}{2}\eta\wedge\mathrm{vol}_T \;=\; \int_{M_3}\frac{w_{(1)}}{2}\sqrt{\det g_{M_3}}\,\mathrm{d}^3x . \qquad (2.57)$$

This played an important role in computing the classical localized Chern-Simons action in [33], which contributes to the field theory partition function on M_3. Using (2.55), (2.56) and (2.57) we see that $A_{(0)}$ in (2.53) is a global one-form if we choose the constant ϱ via

$$\frac{1}{4\varrho}\int_{M_3}\left(w_{(1)}^2 + w_{(2)}\right)\sqrt{\det g_{M_3}}\,\mathrm{d}^3x \;=\; \mathrm{Vol}_\eta . \qquad (2.58)$$

We shall return to this formula in section 2.2.5

2.2.4 Global conformal Kähler structure

Recall that at the beginning of this section we assumed we were given a complete self-dual Einstein metric with Killing vector $\mathcal{K} = \partial_\psi$, of the local form (2.5). We would like to understand when the conformal Kähler structure, studied locally in section 2.1.2, is then globally non-singular. As we shall see, this is not automatically the case. Focusing on the case of toric metrics on a four-ball (all examples in section 2.4 are of this type), with an appropriate restriction on $\mathcal{K}$ we will see that the conformal Kähler structure is indeed everywhere regular. It follows in this case that the Kähler spin^c spinor and instanton $F = \frac{1}{2}\mathcal{R}$ are globally non-singular, and thus that the Killing spinor ϵ given by (2.25) is also globally defined and non-singular. Before embarking on this section, we warn the reader that the discussion is a little involved, and this section is probably better read in conjuction with the explicit examples in section 2.4. In fact the Euclidean AdS_4 metric in section 2.4.1 displays almost all of the generic features we shall encounter.

The self-dual Einstein metrics of section 2.4 are all toric, and we may thus parametrize a choice of toric Killing vector $\mathcal{K}$ as

$$\mathcal{K} \;\; = \;\; b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} \ , \tag{2.59}$$

where we have introduced standard 2π-period coordinates φ_1, φ_2 on the torus $U(1)\times U(1)$. It will be important to fix carefully the orientations here. Since the metrics are defined on a ball, diffeomorphic to $\mathbb{R}^4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ with $U(1) \times U(1)$ acting in the obvious way, we choose ∂_{φ_i} so that the orientations on $\mathbb{R}^2$ induce the given orientation on $\mathbb{R}^4$ (with respect to which the metric has anti-self-dual Weyl tensor). This fixes the relative sign of b_1 and b_2. Given that we have also assumed that $\mathcal{K}$ has no fixed points near the conformal boundary, we must also have b_1 and b_2 non-zero. Thus $b_1/b_2 \in \mathbb{R} \setminus \{0\}$, and its sign will be important in what follows.

Since the self-dual Einstein metric is assumed regular, the one-form $\mathcal{K}$ and its exterior derivative $\mathrm{d}\mathcal{K}$ are both globally defined and regular. The self-dual two-form

$$\Psi \;\; \equiv \;\; (\mathrm{d}\mathcal{K})^+ \;\; \equiv \;\; \frac{1}{2}(\mathrm{d}\mathcal{K} + *\mathrm{d}\mathcal{K}) \ , \tag{2.60}$$

is a *twistor* [86], and the invariant definition of the function/coordinate y in section 2.1 is given in terms of its norm by

$$\frac{2}{y^2} \;\; = \;\; \|\Psi\|^2 \;\; \equiv \;\; \frac{1}{2!}\Psi_{\mu\nu}\Psi^{\mu\nu} \ . \tag{2.61}$$

The complex structure tensor for the conformal Kähler structure is correspondingly

$$J^{\mu}{}_{\nu} \;=\; -y\Psi^{\mu}{}_{\nu}\ , \tag{2.62}$$

where indices are raised and lowered using the self-dual Einstein metric. It is then an algebraic fact that $J^2 = -1$. The conformal Kähler structure will thus be everywhere regular, provided the functions y and $1/y$ are not zero. Of course $y = 0$ is the conformal boundary (which is at infinity, and is not part of the self-dual Einstein space). We are free to choose the sign when taking a square root of (2.61), and without loss of generality we take $y > 0$ in a neighbourhood of the conformal boundary at $y = 0$. Since everything is regular, in particular the norm of the twistor Ψ cannot diverge anywhere (except at infinity), and thus $y \neq 0$ in the interior of the bulk M_4. It follows that y is everywhere positive on M_4.

The Killing vector $\mathcal{K}$ is zero only at the "NUT", namely the fixed origin of $\mathbb{R}^4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$. At this point the two-form $\mathrm{d}\mathcal{K}$, in an orthonormal frame, is a skew-symmetric 4×4 matrix whose weights are precisely the coefficients b_1, b_2 in (2.59).[5] It follows from the definitions (2.60) and (2.61), together with a little linear algebra in such an orthonormal frame, that

$$y_{\mathrm{NUT}} \;=\; \frac{1}{|b_1 + b_2|}\ . \tag{2.63}$$

The conformal Kähler structure will thus be regular everywhere, except potentially where $1/y = 0$. Suppose that $1/y = 0$ at a point $p \in M_4 \setminus \{\mathrm{NUT}\}$. Then $\mathcal{K} = \partial_\psi\,|_p \neq 0$, and thus from the metric (2.5) we see that $1/(By^2)\,|_p \neq 0$. It follows that the function B must tend to zero as $1/y^2$ as one approaches p. We may thus write $B = \frac{c}{y^2} + \mathcal{O}(1/y^3)$, where $c = c(z,\bar{z})$ is non-zero at p. Using the definition of B in terms of w in (2.6) we thus see that $\partial_y w = \frac{2}{y} - \frac{2c}{y^3} + \mathcal{O}(1/y^4)$. There are then various ways to see that the corresponding supersymmetric supergravity solution is *singular*. Perhaps the easiest is to note from the Killing spinor formula (2.25), together with the fact that we may normalize $\zeta^\dagger\zeta = 1$, we have

$$\epsilon^\dagger\epsilon \;=\; \frac{1}{2y}\left(1 + B^{-1}\right)\ , \tag{2.64}$$

which from the above behaviour of B then diverges as we approach the point p. It follows that the Killing spinor ϵ is divergent at p, and the solution is singular.

The solutions are thus singular on $M_4 \setminus \{\mathrm{NUT}\}$ if and only if $\{1/y = 0\} \setminus \{\mathrm{NUT}\}$ is non-empty. Since $y_{\mathrm{NUT}} = 1/|b_1 + b_2|$, the analysis will be a little different for the cases $b_1/b_2 = -1$ and $b_1/b_2 \neq -1$. We thus assume the latter (generic) case

[5] This is perhaps easiest to see by noting that to leading order the metric is flat at the NUT, so one can compute $\mathrm{d}\mathcal{K}$ in an orthonormal frame at the NUT using the flat Euclidean metric on $\mathbb{R}^2 \oplus \mathbb{R}^2$.

for the time being. As in the last paragraph, let us suppose $1/y\mid_p = 0$. Due to the behaviour of B and w near p, it follows from the form of the metric (2.5) that p must lie on one of the axes, *i.e.* at $\rho_1 = 0$ or at $\rho_2 = 0$, where (ρ_i, φ_i) are standard polar coordinates on each copy of $\mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{R}^4 \cong M_4$, $i = 1, 2$.[6] In either case there is then an $S^1 \ni p$ locus of points where $1/y = 0$, as follows by following the orbits of the Killing vector ∂_{φ_2} or ∂_{φ_1}, respectively.

To see when this happens, our analysis will be based on the fact that, since the Killing vector has finite norm in the interior of M_4, one can straightforwardly show that y diverges if and only if $||\mathrm{d}y|| = 0$. It is then convenient to consider the function y restricted to the relevant axis, *i.e.* $y\mid_{\{\rho_1=0\}} \equiv y_2(\rho_2)$ or $y\mid_{\{\rho_2=0\}} \equiv y_1(\rho_1)$. We have $y_1(0) = y_2(0) = y_{\mathrm{NUT}} > 0$. Suppose that $y_i(\rho)$ (for either $i = 1, 2$) starts out decreasing along the axis as we move away from the NUT. Then in fact it must remain monotonic decreasing along the whole axis, until it reaches $y = 0$ at conformal infinity where $\rho = \infty$. The reason for this is simply that if $y_i(\rho)$ has a turning point then[7] $\mathrm{d}y = 0$, which we have already seen can happen only where y diverges, but this contradicts the fact that $y_i(\rho)$ is decreasing from a positive value at $\rho = 0$ (and is bounded below by 0). On the other hand, suppose that $y_i(\rho)$ starts out increasing at the NUT. Then since at conformal infinity $y_i(\infty) = 0$, it follows that $y_i(\rho)$ must have a turning point at some finite $\rho > 0$. At such a point y will diverge, and from our above discussion the solution is singular.

This shows that the key is to examine $\mathrm{d}y$ at the NUT itself. Recall that the coordinate y is a Hamiltonian function for the Killing vector $\mathcal{K}$, *i.e.* $\mathrm{d}y = -\mathcal{K} \lrcorner \omega$. From (2.62), we also know that ω is related to the two-form $\Psi = (\mathrm{d}\mathcal{K})^+$ by $\omega = -y^3\Psi$, yielding $\mathrm{d}y = y^3 \mathcal{K} \lrcorner (\mathrm{d}\mathcal{K})^+$. At the NUT we may again use the polar coordinates (ρ_i, φ_i) for the two copies of $\mathbb{R}^2$, where the metric is to leading order the metric on flat space. In the usual orthonormal frame for these polar coordinates, using the above formulae we then compute to leading order

$$(\mathrm{d}y)|_{\mathrm{NUT}} \simeq \begin{pmatrix} -\frac{b_1}{(b_1+b_2)^2}\,\mathrm{sign}(b_1+b_2)\rho_1 \\ 0 \\ -\frac{b_2}{(b_1+b_2)^2}\,\mathrm{sign}(b_1+b_2)\rho_2 \\ 0 \end{pmatrix} . \tag{2.65}$$

Thus when $b_1/b_2 > 0$ we see that $y_i(\rho)$ starts out decreasing at the NUT, for *both* $i = 1, 2$, and from the previous paragraph it follows that the solution is then globally non-singular! On the other hand, the case $b_1/b_2 < 0$ splits further into two subcases. For simplicity let us describe the case where $b_2 > 0$ (with the case $b_2 < 0$ being similar). Then when $b_1/b_2 < -1$ we have $y_2(\rho)$ starts out increasing at the NUT,

[6] Notice that when $b_1/b_2 = -1$ in fact $1/y = 0$ at the NUT itself, $\rho_1 = \rho_2 = 0$.

[7] Notice that $\mathrm{d}y$ necessarily points along the axis, given the form of the metric (2.5).

which then leads to a singularity along the axis $\rho_1 = 0$ at some finite value of ρ_2; on the other hand, when $-1 < b_1/b_2 < 0$ we have that $y_1(\rho)$ starts out increasing at the NUT, which then leads to a singularity along the axis $\rho_2 = 0$ at some finite value of ρ_1. Notice these two subcases meet where $b_1/b_2 = -1$, when we know that $1/y = 0$ at the NUT itself, $\rho_1 = \rho_2 = 0$.

This leads to the simple picture that all solutions with $b_1/b_2 > 0$ are globally regular, while all solutions with $b_1/b_2 < 0$ are singular, *except* when $b_1/b_2 = -1$. In this latter case y is infinity at the NUT. As one moves out along either axis, y is then necessarily monotonically decreasing to zero by similar arguments to those above. Thus the $b_1/b_2 = -1$ solution is in fact also non-singular, although qualitatively different from the solutions with $b_1/b_2 > 0$. One can show that, regardless of the values of b_1 and b_2, the complex structure (2.62) is always the standard complex structure on flat space at the NUT, meaning that when $b_1/b_2 > 0$ the induced complex structure at the NUT is $\mathbb{C}^2$, while when $b_1/b_2 = -1$ the NUT becomes a point at infinity in the conformal Kähler metric, with the Kähler metric being asymptotically Euclidean. In particular the instanton is zero at the NUT in this case, and so is regular there.

Notice that, for the regular solutions, since $\mathcal{K}$ is nowhere zero away from the NUT we may deduce that also $\mathrm{d}y = -\mathcal{K} \lrcorner \omega$ is nowhere zero (as ω is a global symplectic form on $M_4 \setminus \{\mathrm{NUT}\}$). In particular, y is a global Hamiltonian function for $\mathcal{K}$, and in particular it is a Morse-Bott function on M_4. This implies that y has no critical points on $M_4 \setminus \{\mathrm{NUT}\}$, and thus that y_{NUT} is the *maximum* value of y on M_4. Moreover, the Morse-Bott theory tells us that constant y surfaces on $M_4 \setminus \{\mathrm{NUT}\}$ are all diffeomorphic to $M_3 \cong S^3$.

We shall see all of the above behaviour very explicitly in section 2.4 for the case when the self-dual Einstein metric is simply Euclidean AdS_4. The more complicated Einstein metrics in that section of course also display these features, although the corresponding formulae become more difficult to make completely explicit as the examples become more complicated.

2.2.5 Toric formulae

In this section we shall obtain some further formulae, valid for any toric self-dual Einstein metric on the four-ball. These will be useful for computing the holographic free energy in the next section.

We first note that for $M_3 \cong S^3$ with Reeb vector (2.37) the almost contact volume in (2.57) may be computed using equivariant localization to give

$$\mathrm{Vol}_\eta \;=\; \int_{M_3} \eta \wedge \mathrm{d}\eta \;=\; -\frac{(2\pi)^2}{b_1 b_2}\,. \tag{2.66}$$

This formula also appeared in [33]. One proves (2.66) by an analogous computation to the Duistermaat-Heckman formula in [87]. Specifically, we define a two-form

$$\tilde{\omega} \equiv \tfrac{1}{2}\mathrm{d}(\mathsf{r}^2\eta) , \tag{2.67}$$

on M_4, where r is a choice of radial coordinate with the NUT at $\mathsf{r} = 0$ and the conformal boundary at $\mathsf{r} = \infty$, and notice that

$$\mathrm{Vol}_\eta = -\int_{M_4} \mathrm{e}^{-\mathsf{r}^2/2}\frac{1}{2!}\tilde{\omega}\wedge\tilde{\omega} . \tag{2.68}$$

The minus sign arises here because the natural orientation on M_3 defined in our set-up is opposite to that on the right hand side of (2.68). Specifically, y is decreasing towards the boundary of M_4, so that $\mathrm{d}y$ points inwards from $M_3 = \partial M_4$, while r is increasing towards the boundary, with $\mathrm{d}\mathsf{r}$ pointing outwards.[8] One then evaluates the right hand side of (2.68) using equivariant localization. Specifically, the integrand is

$$\exp\left[-\frac{\mathsf{r}^2}{2}+\tilde{\omega}\right] , \tag{2.69}$$

which is an equivariantly closed form for $\mathcal{K}$, *i.e.* is closed under $\mathrm{d}+\mathcal{K}\lrcorner$, since $\mathcal{K}\lrcorner\tilde{\omega} = -\mathrm{d}(\frac{\mathsf{r}^2}{2})$. The Berline-Vergne equivariant integration theorem then localizes the integral to the fixed point set of $\mathcal{K}$, and one obtains precisely (2.66), with the b_i appearing as the weights of the action of $\mathcal{K}$ at the NUT.[9]

Finally, let us return to the equation (2.58). In fact there is another interpretation of the constant ϱ, in terms of the charge of the Killing spinor under $\mathcal{K}$. To see this, recall that the solution (2.49) to the three-dimensional Killing spinor equation (2.50) is simply constant in our frame, but that was for the case where the gauge field $A_{(0)}$ is given by (2.52), which as we saw in section 2.2.3 is always in a singular gauge on $M_3 \cong S^3$. The gauge transformation $A_{(0)} \to A_{(0)} + \varrho\mathrm{d}\psi$ that we made in (2.53) to obtain a non-singular gauge implies that the correct global spinor χ has a phase dependence

$$\chi^{\mathrm{global}} = \mathrm{e}^{\mathrm{i}\varrho\psi}\begin{pmatrix}\chi_0\\ \chi_0\end{pmatrix} , \tag{2.70}$$

where χ_0 is a constant complex number. Since the frame is invariant under $\mathcal{K} = \partial_\psi$, we thus deduce that ϱ is precisely the charge of the Killing spinor under $\mathcal{K}$.

On the other hand, the total four-dimensional spinor is constructed from the canonical spinor ζ on the conformal Kähler manifold, via (2.25). Thus ϱ is also the

[8] Notice that we could have avoided this by choosing y to be strictly negative on the interior of M_4, rather than strictly positive.

[9] This is then the Duistermaat-Heckman formula when $\tilde{\omega}$ is a symplectic form, *i.e.* when η is a contact form.

charge of ζ under $\mathcal{K}$. This immediately allows us to write down that

$$|\varrho| \;=\; \frac{|b_1|+|b_2|}{2}\ . \tag{2.71}$$

This formula may be fixed by looking at the behaviour at the NUT, where recall that the complex structure is that of $\mathbb{C}^2$. In terms of complex coordinates $z_1 = |z_1|\mathrm{e}^{\mathrm{i}\psi_1}$, $z_2 = |z_2|\mathrm{e}^{\mathrm{i}\psi_2}$, the Kähler spinor ζ, and hence also our Killing spinor, has charges $\frac{1}{2}$ under each of ∂_{ψ_i}, $i = 1, 2$. However, one must be careful to correctly fix the orientations, which leads to the modulus signs in (2.71). More precisely, for $b_1/b_2 > 0$ the conformal Kähler metric fills the interior of a ball in $\mathbb{C}^2$, while for $b_1/b_2 = -1$ instead it is the exterior – see, for example, the discussion at the end of section 2.4.1.

2.3 Holographic free energy

In this section we compute the regularized holographic free energy for a supersymmetric self-dual asymptotically locally Euclidean AdS solution defined on the four-ball, deriving the remarkably simple formula (1.33).

2.3.1 General formulae

The computation of the holographic free energy follows by now standard holographic renormalization methods [88, 89]. The total on-shell action is

$$S^{\mathrm{ren}}_{\mathrm{sugra}} \;=\; S^{\mathrm{grav}}_{\mathrm{bulk}} + S^F + S^{\mathrm{grav}}_{\mathrm{bdry}} + S^{\mathrm{grav}}_{\mathrm{ct}}\ . \tag{2.72}$$

Here the first two terms are the bulk (Euclidean) supergravity action (2.1)

$$S_{\mathrm{sugra}} \;=\; S^{\mathrm{grav}}_{\mathrm{bulk}} + S^F \;\equiv\; -\frac{1}{16\pi G_4}\int_{M_4} (R + 6 - F_{\mu\nu}F^{\mu\nu})\sqrt{\det g}\,\mathrm{d}^4x\ , \tag{2.73}$$

evaluated on a particular solution with topology M_4. The boundary term $S^{\mathrm{grav}}_{\mathrm{bdry}}$ in (2.72) is the Gibbons-Hawking-York term, required so that the equations of motion (2.2) follow from the bulk action (2.73) for a manifold M_4 with boundary. This action is divergent, but we may regularize it using holographic renormalization. Introducing a cut-off at a sufficiently small value of $y = \delta > 0$, with corresponding hypersurface $\mathcal{S}_\delta = \{y = \delta\} \cong M_3$, we have the following total boundary terms

$$S^{\mathrm{grav}}_{\mathrm{bdry}} + S^{\mathrm{grav}}_{\mathrm{ct}} \;=\; \frac{1}{8\pi G_4}\int_{\mathcal{S}_\delta}\left(-\mathscr{K} + 2 + \tfrac{1}{2}R(h)\right)\sqrt{\det h}\,\mathrm{d}^3x\ . \tag{2.74}$$

Here $R(h)$ is the Ricci scalar of the induced metric h_{ij} on $\mathcal{S}_\delta$, and $\mathscr{K}$ is the trace of the second fundamental form of $\mathcal{S}_\delta$, the latter being the Gibbons-Hawking-York boundary term. It is convenient to rewrite the latter using

$$\int_{\mathcal{S}_\delta} \mathscr{K}\sqrt{\det h}\,\mathrm{d}^3x \;=\; \mathcal{L}_n \int_{\mathcal{S}_\delta} \sqrt{\det h}\,\mathrm{d}^3x \;, \tag{2.75}$$

where $\mathcal{L}_n$ is the Lie derivative along the outward pointing normal vector n to the boundary $\mathcal{S}_\delta$.

2.3.2 The four-ball

In this subsection we evaluate the total free energy (2.72) in the case of a supersymmetric self-dual solution on the four-ball $M_4 \cong B^4 \cong \mathbb{R}^4$.

Gauge field contribution

The contribution from the gauge field to (2.72) is

$$S^F \;=\; \frac{1}{16\pi G_4}\int_{M_4} F_{\mu\nu}F^{\mu\nu}\sqrt{\det g}\,\mathrm{d}^4x \;=\; -\frac{1}{8\pi G_4}\int_{M_4} F\wedge F \;=\; \frac{1}{8\pi G_4}\int_{M_3} A_{(0)}\wedge F_{(0)} \;. \tag{2.76}$$

Here in the second equality we used the anti-self-duality $*_4F = -F$. In the last equality we used the fact that on the four-ball $M_4 = B^4 \simeq \mathbb{R}^4$ the curvature $F = \mathrm{d}A$ is globally exact, and then applied Stokes' theorem with $M_3 = \partial M_4$, recalling that the natural orientation on M_3 is induced from an inward-pointing normal vector, with the conformal boundary at $y = 0$. Notice that the contribution from the gauge field is finite, so for S^F there is no need to introduce the cut-off and take the limit $\lim_{\delta\to 0}\mathcal{S}_\delta$.

It was emphasized above that the gauge field $A_{(0)}$ given by (2.44) is in general only valid locally, and in order for $A_{(0)}$ to be a global one-form, we performed the gauge transformation in $A_{(0)} \to A_{(0)} + \varrho\mathrm{d}\psi$ in (2.53).

From the expression,

$$A_{(0)} \;=\; -\frac{1}{4}w_{(1)}\eta + \Xi + \varrho\mathrm{d}\psi \;, \tag{2.77}$$

with

$$\Xi \;=\; \frac{\mathrm{i}}{4}\big(\partial_z w_{(0)}\mathrm{d}z - \partial_{\bar{z}} w_{(0)}\mathrm{d}\bar{z}\big) \;. \tag{2.78}$$

the integral of (2.76) can now be written

$$\int_{M_3} A_{(0)}\wedge F_{(0)} \;=\; \int_{M_3}\Big(\varrho\eta\wedge\mathrm{d}\Xi + \frac{w_{(1)}^3}{32}\eta\wedge\mathrm{vol}_T - \frac{1}{2}w_{(1)}\eta\wedge\mathrm{d}\Xi\Big) \;, \tag{2.79}$$

where we integrated by parts a term containing $\mathrm{d}w_{(1)}$ from $F_{(0)} = \mathrm{d}A_{(0)}$. Using then

(2.55), (2.56), (2.71), and that $\eta \wedge \mathrm{vol}_T = \sqrt{\det g_{M_3}}\mathrm{d}^3x$, equation (2.76) becomes

$$\begin{aligned} S^F &= -\frac{\pi}{2G_4} \cdot \frac{(|b_1|+|b_2|)^2}{4b_1b_2} + \frac{1}{8\pi G_4}\int_{M_3} \frac{w_{(1)}^3}{32}\sqrt{\det g_{M_3}}\mathrm{d}^3x \\ &\quad -\frac{1}{8\pi G_4}\int_{M_3}\frac{1}{8}\big(w_{(1)}^3 + w_{(1)}w_{(2)}\big)\sqrt{\det g_{M_3}}\mathrm{d}^3x \ . \end{aligned} \tag{2.80}$$

Although the integrals in (2.80) are not evaluated, we will see below that these combine with the other contributions to the free energy (2.72).

Bulk gravity contribution

The contribution to the free energy from the bulk gravity part of the action is

$$S_{\mathrm{bulk}}^{\mathrm{grav}} = -\frac{1}{16\pi G_4}\int_{M_4^\delta}(R+6)\,\mathrm{vol}_4 = \frac{1}{16\pi G_4}\int_{M_4^\delta} 6\ \mathrm{vol}_4 \ , \tag{2.81}$$

where we used that on-shell $R = -12$. Here M_4^δ indicates that we have introduced a cut-off along the boundary $\mathcal{S}_\delta = \{y = \delta\} \simeq M_3$, which is necessary as the volume is divergent. The volume form of interest is

$$\mathrm{vol}_4 = \frac{1}{y^4}\mathrm{d}y \wedge (\mathrm{d}\psi + \phi) \wedge B\mathrm{e}^w 2\mathrm{i}\mathrm{d}z \wedge \mathrm{d}\bar{z} \ . \tag{2.82}$$

A computation reveals that this may be written as the exact form

$$-3\,\mathrm{vol}_4 = \mathrm{d}\mathsf{W} \ , \tag{2.83}$$

where we have defined the three-form

$$\mathsf{W} \equiv \frac{1}{2y^2}(\mathrm{d}\psi + \phi)\wedge \mathrm{d}\phi + \frac{1}{y^3}(\mathrm{d}\psi+\phi)\wedge B\mathrm{e}^w 2\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \ . \tag{2.84}$$

We may then integrate over M_4^δ using Stokes' theorem. To do this let us define r to be geodesic distance from the NUT – the origin of $M_4 \cong B^4 \cong \mathbb{R}^4$ that is fixed by the Killing vector $\mathcal{K} = \partial_\psi$. We then more precisely cut off the space also at small $\mathsf{r} > 0$ and let $\mathsf{r} \to 0$, so that we are integrating over $M_4^{\delta,\mathsf{r}}$. The form W may be written

$$\mathsf{W} = \frac{1}{2y^2}(\mathrm{d}\psi+\phi)\wedge\mathrm{d}\phi + \frac{1}{y^3}(\mathrm{d}\psi+\phi)\wedge\omega \ , \tag{2.85}$$

where ω is the conformal Kähler form. As argued in section 2.2.4, when y_{NUT} is finite ω is everywhere a smooth two-form, and thus in particular in polar coordinates near the NUT at $\mathsf{r} = 0$ it takes the form $\omega \simeq \mathsf{r}\mathrm{d}\mathsf{r}\wedge\beta_1 + \mathsf{r}^2\beta_2$ to leading order, where β_1 and β_2 are pull-backs of smooth forms on the $S^3 = S^3_{\mathrm{NUT}}$ at constant $\mathsf{r} > 0$. Because of this, the second term in (2.85) does not contribute to the integral around the NUT.

However, notice that

$$\int_{S^3_{\mathrm{NUT}}} (\mathrm{d}\psi + \phi) \wedge \mathrm{d}\phi \;=\; \int_{M_3^{y=0}} (\mathrm{d}\psi + \phi) \wedge \mathrm{d}\phi \;=\; -\frac{(2\pi)^2}{b_1 b_2} \;, \tag{2.86}$$

follows from a simple application of Stokes' theorem[10], where we have used the almost contact volume (2.66). Using the fact (2.63) that $y_{\mathrm{NUT}} = 1/|b_1 + b_2|$ one thus obtains

$$\int_{M_4^\delta} \mathrm{vol}_4 \;=\; \frac{(2\pi)^2 |b_1 + b_2|^2}{6 b_1 b_2} + \int_{M_3^{y=0}} \left[\frac{1}{3\delta^3} + \frac{w_{(1)}}{4\delta^2}\right] \sqrt{\det g_{M_3}}\, \mathrm{d}^3 x \;, \tag{2.87}$$

so that

$$\begin{aligned} S^{\mathrm{grav}}_{\mathrm{bulk}} \;=\;& \frac{\pi}{2G_4} \cdot \frac{|b_1 + b_2|^2}{2 b_1 b_2} + \frac{1}{8\pi G_4} \cdot \frac{1}{\delta^3} \int_{M_3^{y=0}} \sqrt{\det g_{M_3}}\, \mathrm{d}^3 x \\ & + \frac{3}{32\pi G_4} \cdot \frac{1}{\delta^2} \int_{M_3^{y=0}} w_{(1)} \sqrt{\det g_{M_3}}\, \mathrm{d}^3 x \;. \end{aligned} \tag{2.88}$$

To obtain this result we used the identity

$$\int_{M_3} \left(w_{(1)}^3 + 3 w_{(1)} w_{(2)} + w_{(3)}\right) \sqrt{\det g_{M_3}}\, \mathrm{d}^3 x \;=\; 0 \;, \tag{2.89}$$

which arises from Taylor expanding the Toda equation (2.8) as

$$\begin{aligned} 0 \;=\;& \partial_z \partial_{\bar{z}} w_{(0)} + \mathrm{e}^{w_{(0)}} \left(w_{(1)}^2 + w_{(2)}\right) \\ & + y \left[\partial_z \partial_{\bar{z}} w_{(1)} + \mathrm{e}^{w_{(0)}} \left(w_{(1)}^3 + 3 w_{(1)} w_{(2)} + w_{(3)}\right)\right] + \mathcal{O}(y^2) \;. \end{aligned} \tag{2.90}$$

Because $w_{(1)}$ is a smooth global function on M_3, the second line implies (2.89) after integrating over the boundary and using Stokes' theorem.

It remains to evaluate the boundary terms $S^{\mathrm{grav}}_{\mathrm{bdry}} + S^{\mathrm{grav}}_{\mathrm{ct}}$. After a computation, and again using (2.89), one obtains

$$\begin{aligned} S^{\mathrm{grav}}_{\mathrm{bdry}} + S^{\mathrm{grav}}_{\mathrm{ct}} \;=\;& -\frac{1}{8\pi G_4 \delta^3} \int_{M_3^{y=0}} \sqrt{\det g_{M_3}}\, \mathrm{d}^3 x - \frac{3}{32\pi G_4 \delta^2} \int_{M_3^{y=0}} w_{(1)} \sqrt{\det g_{M_3}}\, \mathrm{d}^3 x \\ & + \frac{1}{256\pi G_4} \int_{M_3} \left(3 w_{(1)}^3 + 4 w_{(1)} w_{(2)}\right) \sqrt{\det g_{M_3}}\, \mathrm{d}^3 x \;. \end{aligned} \tag{2.91}$$

Adding (2.91) to the bulk gravity term (2.88) we see that the divergent terms do indeed precisely cancel, and further combining with (2.80) we see that the terms involving the integrals of $w_{(i)}$ also all cancel.

The computations we have done are valid only for globally regular solutions, and recall these divide into the two cases $b_1/b_2 > 0$, and $b_1/b_2 = -1$. In the first case

[10] This follows since $\mathrm{d}[(\mathrm{d}\psi + \phi) \wedge \mathrm{d}\phi] = 0$.

the first term in (2.80) combines with the first term in (2.88) to give

$$S^{\mathrm{ren}}_{\mathrm{sugra}} \;=\; \frac{\pi}{2G_4}\cdot\frac{(|b_1|+|b_2|)^2}{4|b_1 b_2|}\ , \tag{2.92}$$

where notice $|b_1+b_2| = |b_1|+|b_2|$. On the other hand the isolated case with $b_1/b_2=-1$ has $b_1+b_2=0$, so that the free energy comes entirely from the first term in (2.80), which remarkably is then also given by the formula (2.92). Thus for all regular supersymmetric solutions we have shown that (2.92) holds, which is the result advertised in (1.33) in the introduction.

2.4 Examples

In this section we illustrate our general results by discussing three explicit families of solutions. These consist of three sets of self-dual Einstein metrics on the four-ball, studied previously by some of the authors in [61–64]. We begin with simply AdS_4 in section 2.4.1. Although the metric is trivial, the one-parameter family of instantons given by our general results is non-trivial, and it turns out that this family is identical to that in [61]. The solutions in sections 2.4.2 and 2.4.3 each add a deformation parameter, meaning that the metrics in each subsequent section generalize that in the previous section. *Particular* supersymmetric instantons on these backgrounds were found in [62–64], but our general results allow us to study the most general choice of instanton, leading to new solutions. Furthermore, in section 2.4.4 we indicate how to generalize these metrics further by adding an *arbitrary* number of parameters. Towards the end of this section, Figure 2.1 then summarizes the connection between all the metric studied in this chapter.

2.4.1 AdS_4

The metric on Euclidean AdS_4 can be written as

$$\mathrm{d}s^2_{\mathrm{EAdS}_4} \;=\; \frac{\mathrm{d}q^2}{q^2+1}+q^2\left(\mathrm{d}\vartheta^2+\cos^2\vartheta\mathrm{d}\varphi_1^2+\sin^2\vartheta\mathrm{d}\varphi_2^2\right)\ . \tag{2.93}$$

Here q is a radial variable with $q\in[0,\infty)$, so that the NUT is at $q=0$ while the conformal boundary is at $q=\infty$. The coordinate $\vartheta\in[0,\frac{\pi}{2}]$, with the endpoints being the two axes of $\mathbb{R}^2\oplus\mathbb{R}^2\cong\mathbb{R}^4$. The AdS_4 metric is of course both self-dual and anti-self-dual.

Writing a general choice of Reeb vector field as $\mathcal{K}=b_1\partial_{\varphi_1}+b_2\partial_{\varphi_2}$, as in our general discussion (2.59), the function y is then defined in terms of $\mathcal{K}$ via (2.60) and

(2.61). Using these formulae one easily computes

$$y(q,\vartheta) \;=\; \frac{1}{\sqrt{(b_2+b_1\sqrt{q^2+1})^2\cos^2\vartheta+(b_1+b_2\sqrt{q^2+1})^2\sin^2\vartheta}}\;. \tag{2.94}$$

Notice that indeed $y_{\mathrm{NUT}} = 1/|b_1+b_2|$, in agreement with (2.63). Using (2.94) one can also verify the general behaviour in section 2.2.4 very explicitly. In particular, we see the very different global behaviour, depending on the sign of b_1/b_2. If $b_1/b_2 > 0$ then $1/y$ is nowhere zero, while if $b_1/b_2 < 0$ instead $1/y$ has a zero on M_4. More precisely, if $-1 < b_1/b_2 < 0$ then $1/y = 0$ at $\{\vartheta = 0, q = \sqrt{b_2^2 - b_1^2}/|b_1|\}$, while if $b_1/b_2 < -1$ then $1/y = 0$ at $\{\vartheta = \frac{\pi}{2}, q = \sqrt{b_1^2 - b_2^2}/|b_2|\}$. These are each a copy of S^1 at one or the other of the "axes" of $\mathbb{R}^2 \oplus \mathbb{R}^2$, at the corresponding radius given by q. In the special case that $b_1 = -b_2$ we have $1/y = 0$ at the NUT itself, where the axes meet. These comments of course all agree with the general analysis in section 2.2.4, except here all formulae can be made completely explicit.

We thus indeed obtain smooth solutions for all $b_1/b_2 > 0$, as well as the isolated non-singular solution with $b_1/b_2 = -1$. In fact it is not difficult to check that the former are precisely the solutions first found in [61], where the parameter $b^2 = b_2/b_1$ (compare to the formulae at the beginning of section 2.5 of [61]). To see this we may compute the instanton using the formulae in section 2.1, finding

$$A \;=\; \frac{\left(b_1+b_2\sqrt{q^2+1}\right)\mathrm{d}\varphi_1+\left(b_2+b_1\sqrt{q^2+1}\right)\mathrm{d}\varphi_2}{2\sqrt{(b_2+b_1\sqrt{q^2+1})^2\cos^2\vartheta+(b_1+b_2\sqrt{q^2+1})^2\sin^2\vartheta}}\;, \tag{2.95}$$

which agrees with the corresponding formula in [61]. In particular, one can check that this gives a regular instanton when $b_1/b_2 > 0$, with the particular cases that $b_1/b_2 = \pm 1$ giving a *trivial* instanton, and correspondingly the conformal Kähler structure is flat. We shall comment further on this below. Moreover, one can also check that the singular instantons with $b_1/b_2 < 0$ are singular at precisely the locus that $1/y = 0$, again in agreement with our general discussion.

In this case we may also compute all other functions appearing in sections 2.1, 2.2 and 2.3 very explicitly. For example, we find

$$B(q,\vartheta) \;=\; \frac{(b_2+b_1\sqrt{q^2+1})^2\cos^2\vartheta+(b_1+b_2\sqrt{q^2+1})^2\sin^2\vartheta}{q^2(b_1^2\cos^2\vartheta+b_2^2\sin^2\vartheta)}\;, \tag{2.96}$$

while the functions $w_{(1)}$ and $w_{(2)}$ on $\partial M_4 = M_3 \cong S^3$ appearing in the free energy computations are given by

$$w_{(1)} \;=\; \frac{-4b_1b_2}{\sqrt{b_1^2\cos^2\vartheta+b_2^2\sin^2\vartheta}}\;,\quad w_{(2)} \;=\; \frac{-2\left(3b_1^2b_2^2+b_1^4\cos^2\vartheta+b_2^4\sin^2\vartheta\right)}{b_1^2\cos^2\vartheta+b_2^2\sin^2\vartheta} \tag{2.97}$$

Using these expressions one can verify all of the key formulae in our general analysis very explicitly. For example, the integrals in (2.66), (2.80), (2.88) and (2.91) are all easily computed in closed form.

Finally, let us return to discuss the special cases $b_1/b_2 = \pm 1$, where recall that the instanton is trivial and the conformal Kähler structure is flat. The latter is thus locally the flat Kähler metric on $\mathbb{C}^2$, but in fact in the two cases $b_1/b_2 = \pm 1$ the Euclidean AdS$_4$ metric is conformally embedded into *different* regions of $\mathbb{C}^2$. Notice this has to be the case, because the conformal factor y of the $b_1/b_2 = +1$ solution has $y_{\rm NUT} = 1/(2|b_1|)$, while for the $b_1/b_2 = -1$ solution instead $y_{\rm NUT} = \infty$. We may see this very concretely by writing the flat Kähler metric on $\mathbb{C}^2$ as

$$\mathrm{d}s^2_{\rm flat} = \mathrm{d}R^2 + R^2\left(\mathrm{d}\vartheta^2 + \cos^2\vartheta \mathrm{d}\varphi_1^2 + \sin^2\vartheta \mathrm{d}\varphi_2^2\right) . \tag{2.98}$$

In both cases the change of radial coordinate to (2.93) is

$$q(R) = \frac{2R}{|R^2-1|} . \tag{2.99}$$

However, for the $b_1/b_2 = +1$ case the range of R is $0 \leq R < 1$, with the NUT being at $R = 0$ and the conformal boundary being at $R = 1$; while for the $b_1/b_2 = -1$ case the range of R is instead $1 < R \leq \infty$, with the NUT being at $R = \infty$ (and the conformal boundary again being at $R = 1$). In particular the two conformal factors are

$$y(R) = \tfrac{1}{2|b_1|}|R^2 - 1| . \tag{2.100}$$

The two solutions $b_1/b_2 = \pm 1$ thus effectively fill opposite sides of the unit sphere in $\mathbb{C}^2$, and because of this they induce opposite orientations on S^3. Again, this may be seen rather explicitly in various formulae. For example, $w_{(1)} = \mp 4|b_1|$ in the two cases, so that the boundary Killing spinor equation (2.50) on the round S^3 becomes respectively $\nabla_i^{(3)}\chi = \mp\frac{\mathrm{i}}{2}|b_1|\gamma_i\chi$.

2.4.2 Taub-NUT-AdS$_4$

The Taub-NUT-AdS$_4$ metrics are a one-parameter family of self-dual Einstein metrics on the four-ball, and have been studied in detail in [62,63]. The metric may be written as

$$\mathrm{d}s^2_4 = \frac{r^2-s^2}{\Omega(r)}\mathrm{d}r^2 + (r^2-s^2)(\tau_1^2+\tau_2^2) + \frac{4s^2\Omega(r)}{r^2-s^2}\tau_3^2 , \tag{2.101}$$

where

$$\Omega(r) = (r\mp s)^2[1+(r\mp s)(r\pm 3s)] , \tag{2.102}$$

and τ_1, τ_2, τ_3 are left-invariant one-forms on $SU(2) \simeq S^3$. The latter may be written in terms of Euler angular variables as

$$\tau_1 + \mathrm{i}\tau_2 \;=\; \mathrm{e}^{-\mathrm{i}\varsigma}(\mathrm{d}\theta + \mathrm{i}\sin\theta \mathrm{d}\varphi) \;, \qquad \tau_3 \;=\; \mathrm{d}\varsigma + \cos\theta \mathrm{d}\varphi \;. \tag{2.103}$$

Here ς has period 4π, while $\theta \in [0, \pi]$ with φ having period 2π. The radial coordinate r lies in the range $r \in [s, \infty)$, with the NUT (origin of the ball $\cong \mathbb{R}^4$) being at $r = s$. The parameter $s > 0$ is referred to as the *squashing parameter*, with $s = \frac{1}{2}$ being the Euclidean AdS$_4$ metric studied in the previous section. Indeed, the metric is asymptotically locally Euclidean AdS as $r \to \infty$, with

$$\mathrm{d}s_4^2 \;\approx\; \frac{\mathrm{d}r^2}{r^2} + r^2(\tau_1^2 + \tau_2^2 + 4s^2\tau_3^2) \;, \tag{2.104}$$

so that the conformal boundary at $r = \infty$ is a biaxially squashed S^3.

Using the results of this chapter we may write a general choice of Reeb vector field as $\mathcal{K} = (b_1 + b_2)\partial_\varphi + (b_1 - b_2)\partial_\varsigma$, as in our general discussion (2.59), and the function y is then defined in terms of $\mathcal{K}$ via (2.60) and (2.61). Using these one computes

$$\begin{aligned}\frac{1}{y(r,\theta)^2} \;&=\; [2(b_1 - b_2)(r - s)s + (b_1 + b_2)(1 + 2(r - s)s)\cos\theta]^2 \\ &\quad + (b_1 + b_2)^2 [1 + (r - s)(r + 3s)] \sin^2\theta \;.\end{aligned} \tag{2.105}$$

Notice that indeed $y_{\mathrm{NUT}} = \lim_{r\to s} y(r, \theta) = 1/|b_1 + b_2|$. We see that if $b_1/b_2 > 0$ or $b_1/b_2 = -1$ then $1/y$ is indeed never zero (except at the NUT in the latter case), as expected. In this way we obtain a *two-parameter* family of regular supersymmetric solutions, parametrized by the squashing parameter s and b_1/b_2. One can also compute explicitly the corresponding instanton F for a general choice of s and b_1/b_2, although in practice it turns out to be more convenient to derive this as a special limit of the Plebański-Demiański solutions, discussed in section 2.4.3. This is shown in the appendix of [1]. In the remainder of this subsection we shall instead discuss further some special cases, making contact with the previous results [62, 63].

While the Taub-NUT-AdS metric (2.101) has $SU(2) \times U(1)$ isometry, a generic choice of the Killing vector $\mathcal{K} = (b_1 + b_2)\partial_\varphi + (b_1 - b_2)\partial_\varsigma$ breaks the symmetry of the full solution to $U(1) \times U(1)$. In particular, this symmetry is also broken by the corresponding instanton A. On the other hand, in [62, 63] the $SU(2) \times U(1)$ symmetry of the metric was *also* imposed on the gauge field, which results in two one-parameter subfamilies of the above two-parameter family of solutions, which are 1/4 BPS and 1/2 BPS, respectively. In each case this effectively fixes the Killing vector $\mathcal{K}$ (or rather the parameter b_1/b_2) as a function of the squashing parameter s.

1/4 BPS solution: This solution is simple enough that it can be presented in complete detail. The coordinate transformation to the (2.5) form for the 1/4 BPS solution reads

$$r - s \;=\; 1/y \;, \qquad -\,2s\tau_3 \;=\; \mathrm{d}\psi + \phi \;, \tag{2.106}$$

and

$$y^2(r^2 - s^2) \;=\; \mathrm{e}^w B(1 + |z|^2)^2 \;, \qquad \frac{r^2 - s^2}{\Omega(r)} \;=\; y^2 B \;. \tag{2.107}$$

Notice immediately that at the NUT $r = s$ we have $1/y = 0$, so that this solution must have $b_1 = -b_2$, as we shall find explicitly below. The metric $(\tau_1^2 + \tau_2^2)$ is diffeomorphic to the Fubini-Study metric on $\mathbb{CP}^1 \cong S^2$:

$$\tau_1^2 + \tau_2^2 \;=\; \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(1 + |z|^2)^2} \;. \tag{2.108}$$

The metric functions then simplify to

$$B(y) \;=\; \frac{1 + 2sy}{1 + 4sy + y^2} \;, \qquad w(y, z, \bar{z}) \;=\; \log \frac{1 + 4sy + y^2}{(1 + |z|^2)^2} \;, \tag{2.109}$$

and it is straightforward to check these satisfy the defining equation (2.6) and Toda equation (2.8). The conformally related scalar-flat Kähler metric is

$$\mathrm{d}s^2_{\text{Kähler}} \;=\; \frac{1 + 2sy}{1 + 4sy + y^2}\mathrm{d}y^2 + (1 + 2sy)(\tau_1^2 + \tau_2^2) + \frac{4s^2(1 + 4sy + y^2)}{1 + 2sy}\tau_3^2 \;, \tag{2.110}$$

with Kähler form

$$\omega \;=\; -\mathrm{d}y \wedge 2s\tau_3 + (1 + 2sy)\tau_1 \wedge \tau_2 \;=\; -\mathrm{d}\left[(1 + 2sy)\tau_3\right] \;. \tag{2.111}$$

Using the formula (2.9) for the gauge field A, we compute

$$A \;=\; \frac{1}{2}(4s^2 - 1)\frac{r - s}{r + s}\tau_3 + \text{pure gauge} \;, \tag{2.112}$$

which we see reproduces the 1/4 BPS choice of instanton in section 3.3 of [63].[11] The supersymmetric Killing vector is $\mathcal{K} = \partial_\psi = -\frac{1}{2s}\partial_\varsigma$ and so generates the Hopf fibration of S^3. Since $\varsigma = \varphi_1 - \varphi_2$, $\varphi = \varphi_1 + \varphi_2$ we hence find

$$b_1 \;=\; -b_2 \;=\; -\frac{1}{4s} \;, \tag{2.113}$$

[11] Notice that in [63] the opposite orientation convention was chosen, so that that instanton in [63] is self-dual, rather than anti-self-dual. Recall also from the discussion above equation (2.9) that the overall sign of the instanton is correlated with the sign of the supersymmetric Killing vector $\mathcal{K}$. Here $\mathcal{K} = -\frac{1}{2s}\partial_\varsigma$, which is minus the expression in [63], hence leading to the opposite sign for the instanton gauge field A.

which using (2.92) yields the holographic free energy

$$S^{\mathrm{ren}}_{\mathrm{sugra}} = \frac{\pi}{2G_4}. \tag{2.114}$$

This formula matches the result of section 5.4 of [63].

1/2 BPS solution: The Taub-NUT-AdS metric (2.101) also admits a 1/2 BPS solution [62,63]. We hence have two linearly independent Killing spinors, which may be parametrized by an arbitrary choice of constant two-component spinor $\chi_{(0)} = \begin{pmatrix} \mathsf{p} \\ \mathsf{q} \end{pmatrix} \in \mathbb{C}^2 \setminus \{0\}$.[12] The correspondong Killing vector is given by the unlikely expression

$$\begin{aligned} \mathcal{K} &= (2s+\sqrt{4s^2-1})\Big[2\mathrm{Im}\,[\mathrm{e}^{\mathrm{i}\varphi}\mathsf{p}\bar{\mathsf{q}}]\partial_\theta + \left(|\mathsf{p}|^2-|\mathsf{q}|^2+2\mathrm{Re}\,[\mathrm{e}^{\mathrm{i}\varphi}\mathsf{p}\bar{\mathsf{q}}]\cot\theta\right)\partial_\varphi\Big] \\ &+ \Big[(|\mathsf{p}|^2+|\mathsf{q}|^2)\left(\tfrac{1}{2s}-2s-\sqrt{4s^2-1}\right) - 2\mathrm{Re}\,[\mathrm{e}^{\mathrm{i}\varphi}\mathsf{p}\bar{\mathsf{q}}](2s+\sqrt{4s^2-1})\csc\theta\Big]\,\partial_\varsigma\,. \end{aligned} \tag{2.115}$$

Since multiplying $\chi_{(0)}$ by a non-zero complex number $\lambda \in \mathbb{C}^*$ simply rescales $\mathcal{K}$ by $|\lambda|^2$, this leads to a $\mathbb{CP}^1$ family of choices of Killing vector $\mathcal{K}$ in this case. Of course, the vector (2.115) is not toric for generic choice of $\chi_{(0)}$. Nevertheless, one can still compute the various geometric quantities in section 2.1. In particular one can check that the formula (2.19) for the instanton gives

$$A = s\sqrt{4s^2-1}\frac{r-s}{r+s}\tau_3 + \text{pure gauge}\,, \tag{2.116}$$

for any choice of $\mathcal{K}$ in (2.115), which agrees with the expression in [62, 63]. Notice that the instanton is invariant under the $SU(2)\times U(1)$ symmetry of the metric, even though a choice of Killing vector $\mathcal{K}$ breaks this symmetry. Indeed, in this case the conformal factor $y = y(r,\theta)$ for toric solutions given by (2.105) depends non-trivially on both r and θ, thus also breaking the $SU(2)$ symmetry of the underlying Taub-NUT-AdS metric. This is to be contrasted with the 1/4 BPS solution, where instead (2.105) reduces simply to $y = y(r) = 1/(r-s)$ (see (2.106)).

The toric choices of $\mathcal{K}$ for these 1/2 BPS solutions correspond to the poles of the $\mathbb{CP}^1$ parameter space. For example, choosing $\mathsf{p}=1$, $\mathsf{q}=0$ above gives

$$\mathcal{K} = \left(2s+\sqrt{4s^2-1}\right)\partial_\varphi + \left(\tfrac{1}{2s}-2s-\sqrt{4s^2-1}\right)\partial_\varsigma\,, \tag{2.117}$$

so that

$$b_1 = \frac{1}{4s}\,, \qquad b_2 = -\frac{1}{4s}+2s+\sqrt{4s^2-1}\,. \tag{2.118}$$

[12]The full Killing spinor is given by substituting this into the right hand side of (2.29) of [63].

The free energy (2.92) is thus

$$S^{\text{ren}}_{\text{sugra}} \;=\; \frac{2\pi s^2}{G_4}\ , \tag{2.119}$$

which of course matches the result obtained in section 4.4 of [63].

2.4.3 Plebański-Demiański

The Taub-NUT-AdS metric has been extended to a two-parameter family of smooth self-dual Einstein metrics on the four-ball in [64], which lie in the Plebański-Demiański class of local solutions [90] to Einstein-Maxwell theory. We will henceforth refer to the solution of [64] as "Plebański-Demiański". The metric may be written as

$$\mathrm{d}s^2_{\text{PD}} \;=\; \frac{\mathcal{P}(q)}{q^2-p^2}(\mathrm{d}\tau+p^2\mathrm{d}\sigma)^2-\frac{\mathcal{P}(p)}{q^2-p^2}(\mathrm{d}\tau+q^2\mathrm{d}\sigma)^2+\frac{q^2-p^2}{\mathcal{P}(q)}\mathrm{d}q^2-\frac{q^2-p^2}{\mathcal{P}(p)}\mathrm{d}p^2, \tag{2.120}$$

where

$$\mathcal{P}(x) \;=\; (x-p_1)(x-p_2)(x-p_3)(x-p_4)\ . \tag{2.121}$$

The roots of the quartic $\mathcal{P}(x)$ can be expressed in terms of the two parameters of the solution, $\hat{a}$ and v, as

$$\begin{aligned} p_1 &= -\frac{1}{2}-\sqrt{1+\hat{a}^2-v^2}\ , \quad p_3=\frac{1}{2}-\hat{a}\ , \\ p_2 &= -\frac{1}{2}+\sqrt{1+\hat{a}^2-v^2}\ , \quad p_4=\frac{1}{2}+\hat{a}\ . \end{aligned} \tag{2.122}$$

The coordinate $p\in[p_3,p_4]$ is essentially a polar angle variable, while $q\in[p_4,\infty)$ plays the role of a radial coordinate, with the conformal boundary being at $q=\infty$. The NUT, *i.e.* the origin of $\mathbb{R}^4$, is located at $p=p_3$, $q=p_4$. The Killing vectors ∂_τ, ∂_σ generate the $U(1)\times U(1)$ toric symmetry of the solution, with the coordinates related to our standard 2π-period coordinates φ_1, φ_2 via

$$\begin{aligned} \tau &= \frac{2p_3^2}{\mathcal{P}'(p_3)}\varphi_1-\frac{2p_4^2}{\mathcal{P}'(p_4)}\varphi_2\ , \\ \sigma &= -\frac{2}{\mathcal{P}'(p_3)}\varphi_1+\frac{2}{\mathcal{P}'(p_4)}\varphi_2\ . \end{aligned} \tag{2.123}$$

In order that the metric is smooth on the four-ball, the parameters must obey $v^2>2|\hat{a}|$. The Taub-NUT-AdS metric of the previous subsection is obtain in the limit $\hat{a}\to 0$. Setting further $v=1$, one recovers Euclidean AdS_4.

It is straightforward, but tedious, to express the metric (2.120) in the form (2.5), with an arbitrary choice of toric Killing vector $\mathcal{K}=b_1\partial_{\varphi_1}+b_2\partial_{\varphi_2}$. For the special case of the Killing vector and instanton in the solution of [64], this is done in the appendix of [1].

In the (τ, σ) coordinates an arbitrary Killing vector may be written as

$$\mathcal{K} \;=\; b_\tau \partial_\tau + b_\sigma \partial_\sigma \;, \tag{2.124}$$

where

$$b_\tau \;=\; \frac{2p_3^2}{\mathcal{P}'(p_3)} b_1 - \frac{2p_4^2}{\mathcal{P}'(p_4)} b_2 \;, \qquad b_\sigma \;=\; -\frac{2}{\mathcal{P}'(p_3)} b_1 + \frac{2}{\mathcal{P}'(p_4)} b_2 \;. \tag{2.125}$$

Using (2.60) and (2.61) one can calculate

$$\begin{aligned} \frac{1}{y(p,q)^2} \;=\; & \frac{1}{4} \frac{1}{(q^2-p^2)^2} \Bigg\{ \Bigg[\left(\frac{2\mathcal{P}(q)}{q-p} - \mathcal{P}'(q) \right) (b_\tau + b_\sigma p^2) \\ & - \left(\frac{2\mathcal{P}(p)}{q-p} + \mathcal{P}'(p) \right) (b_\tau + b_\sigma q^2) \Bigg]^2 - 4 b_\sigma^2 \mathcal{P}(q) \mathcal{P}(p) (q+p)^2 \Bigg\} \;. \end{aligned} \tag{2.126}$$

Notice that this is a sum of two non-negative terms. Furthermore, these terms may vanish only when evaluated at the roots $p = p_3$, $p = p_4$ or $q = p_4$, which correspond to the axes of $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$. Let us calculate these limits:

$$\begin{aligned} \lim_{p \to p_3} \frac{1}{y^2} \;&=\; \left(\frac{(b_1+b_2)v^2 + 2\hat{a} b_1 + b_2(2q-1)}{v^2 + 2\hat{a}} \right)^2 \;, \\ \lim_{p \to p_4} \frac{1}{y^2} \;&=\; \left(\frac{(b_1+b_2)v^2 - 2\hat{a} b_2 + b_1(2q-1)}{v^2 - 2\hat{a}} \right)^2 \;, \\ \lim_{q \to p_4} \frac{1}{y^2} \;&=\; \left(\frac{(b_1+b_2)v^2 - 2\hat{a} b_2 + b_1(2p-1)}{v^2 - 2\hat{a}} \right)^2 \;. \end{aligned} \tag{2.127}$$

A careful analysis of the above limits shows that $1/y$ does not vanish, and hence the metric is regular, whenever $b_1/b_2 > 0$, while $1/y = 0$ only at the NUT when $b_1/b_2 = -1$. On the other hand, the the solution is indeed singular if $b_1/b_2 < 0$ and $b_1/b_2 \neq -1$. Notice that we also easily recover the formula (2.63) for the conformal factor at the NUT: $\lim_{p \to p_3,\ q \to p_4} y = 1/|b_1 + b_2|$.

In [64], particular supersymmetric instantons (particular choices of b_1/b_2 for fixed $\hat{a}$ and v) were studied for this two-parameter family of metrics, which by construction lie within the Plebański-Demiański ansatz. The results of this chapter extend these results to a general choice of instanton on the same background, parametrized by b_1/b_2, leading to a *three-parameter* family of regular supersymmetric solutions. The general expression for this instanton is lengthy, but computable, and the interested reader may find the details in the appendix of [1].

2.4.4 Infinite parameter generalization

In each subsection we have generalized the metrics of the previous subsection by adding a parameter, and one might wonder whether one can find more general self-dual Einstein metrics on the four-ball. In [86] the authors studied the general local geometry of toric self-dual Einstein metrics, which thus includes all the solutions (locally) above. In appropriate coordinates the metric takes the form

$$\mathrm{d}s^2_{\mathrm{toric}} = \frac{4\rho^2(\mathcal{W}_\rho^2+\mathcal{W}_\varpi^2)-\mathcal{W}^2}{4\mathcal{W}^2}\mathrm{d}s^2_{\mathcal{H}^2} + \frac{4}{\mathcal{W}^2(4\rho^2(\mathcal{W}_\rho^2+\mathcal{W}_\varpi^2)-\mathcal{W}^2)}\mathrm{d}s^2_2 \ , \quad (2.128)$$

where we have defined

$$y^{\mathrm{can}}(\rho,\varpi) \equiv \sqrt{\rho}\,\mathcal{W}(\rho,\varpi) \ , \quad (2.129)$$

and

$$\begin{aligned} \mathrm{d}s^2_2 &= \left(y^{\mathrm{can}}_\rho \mathrm{d}\nu + (\varpi y^{\mathrm{can}}_\rho - \rho y^{\mathrm{can}}_\varpi)\mathrm{d}\varphi\right)^2 + \left(y^{\mathrm{can}}_\varpi \mathrm{d}\nu + (\rho y^{\mathrm{can}}_\rho + \varpi y^{\mathrm{can}}_\varpi - y^{\mathrm{can}})\mathrm{d}\varphi\right)^2 \ , \\ \mathrm{d}s^2_{\mathcal{H}^2} &= \frac{\mathrm{d}\rho^2+\mathrm{d}\varpi^2}{\rho^2} \ , \end{aligned} \quad (2.130)$$

where $\mathrm{d}s^2_{\mathcal{H}^2}$ is the metric on hyperbolic two-space $\mathcal{H}^2$, regarded as the upper half plane with boundary at $\rho = 0$. We denote partial derivatives as $\mathcal{W}_\rho \equiv \partial_\rho \mathcal{W}$, $y^{\mathrm{can}}_\varpi \equiv \partial_\varpi y^{\mathrm{can}}$, *etc.* The metric (2.128) is entirely determined by the choice of function $\mathcal{W} = \mathcal{W}(\rho,\varpi)$, and the metric is self-dual Einstein if and only if this solves the eigenfunction equation

$$\Delta_{\mathcal{H}^2}\mathcal{W} = \frac{3}{4}\mathcal{W} \qquad \Longleftrightarrow \qquad \mathcal{W}_{\rho\rho}+\mathcal{W}_{\varpi\varpi} = \frac{3}{4\rho^2}\mathcal{W} \ . \quad (2.131)$$

Unlike the Toda equation (2.8) this is linear, and one may add solutions. In particular there is a basic solution

$$\mathcal{W}(\rho,\varpi;\lambda) = \frac{\sqrt{\rho^2+(\varpi-\lambda)^2}}{\sqrt{\rho}} \ , \quad (2.132)$$

where λ is any constant. Via linearity then

$$\mathcal{W}(\rho,\varpi) = \sum_{i=1}^m \alpha_i \mathcal{W}(\rho,\varpi;\lambda_i) \ , \quad (2.133)$$

also solves (2.131), for arbitrary constants α_i, λ_i, $i = 1,\ldots,m$. We refer to (2.133) as an *m-pole solution*. Of course, one could also replace the sum in (2.133) by an integral, smearing the monopoles in some chosen charge distribution.

Thus the *local* construction of toric self-dual Einstein metrics is very straightforward – the above gives an infinite-dimensional space. However, understanding when the above metrics extend to complete asymptotically locally hyperbolic metrics on a ball (or indeed any other topology for M_4) is more involved. In appendix C of [1] it is shown that the general 2-pole solution is simply (Euclidean) AdS_4, while the general 3-pole solution is precisely the two-parameter Plebański-Demiański solutions of section 2.4.3. This requires taking into account the symmetries of (2.128) (in particular the $PSL(2, \mathbb{R})$ symmetry of $\mathcal{H}^2$), and then making a number of rather non-trivial coordinate transformations. We refer to [1] for these details.

Some work has also been done on global properties of the metrics (2.128) in [91], although the focus in that reference was on constructing complete asymptotically locally Euclidean scalar-flat Kähler metrics, which are conformal to (2.128). However, these have non-trivial Lens space boundaries S^3/Γ, and correspondingly the second Betti number $b_2 = \dim H_2(M_4, \mathbb{R})$ of the filling M_4 is non-zero (they contain "bolt S^2s"). The corresponding complete self-dual Einstein metrics in Theorem B of that reference then also do not have the topology of the ball. Thus it remains an interesting open problem to understand when the general m-pole metrics extend to complete metrics on the ball.[13]

Finally, let us remark that in [92] Lebrun has constructed infinitely many self-dual Einstein metrics on the four-ball using twistor methods. This is essentially a deformation argument, where one starts with (the twistor space of) Euclidean AdS_4, and perturbs the twistor space. However, as such this is rather more implicit than the toric metrics above, and in order to construct supersymmetric solutions one needs to ensure that the resulting self-dual Einstein metric has at least one Killing vector field. Nevertheless, this might be an alternative method for analyzing regularity of the above m-pole solutions, at least in a neighbourhood of Euclidean AdS_4 in parameter space.

2.5 Conclusions

The main result of this chapter is the proof of the formula (2.92) for the holographically renormalized on-shell action in minimal four-dimensional supergravity. Moreover, we discussed the construction of regular supersymmetric solutions of this theory[14], based on self-dual Einstein metrics on the four-ball equipped with a one-parameter family of instanton fields for the graviphoton. Specifically, if the self-dual Einstein metric admits n parameters, our construction produces an $(n+1)$-parameter family of solutions. We have shown that the renormalized on-shell action does *not* depend on the n metric parameters, but only on this last "instanton parameter". This matches beautifully the field theory results of [33].

[13] At the end of reference [86] it is briefly noted that one can obtain regular m-pole metrics by deforming, for example, a given 3-pole solution. It would be interesting to examine the details of this deformation argument further.

[14] These uplift to solutions of eleven-dimensional supergravity using the results of [93].

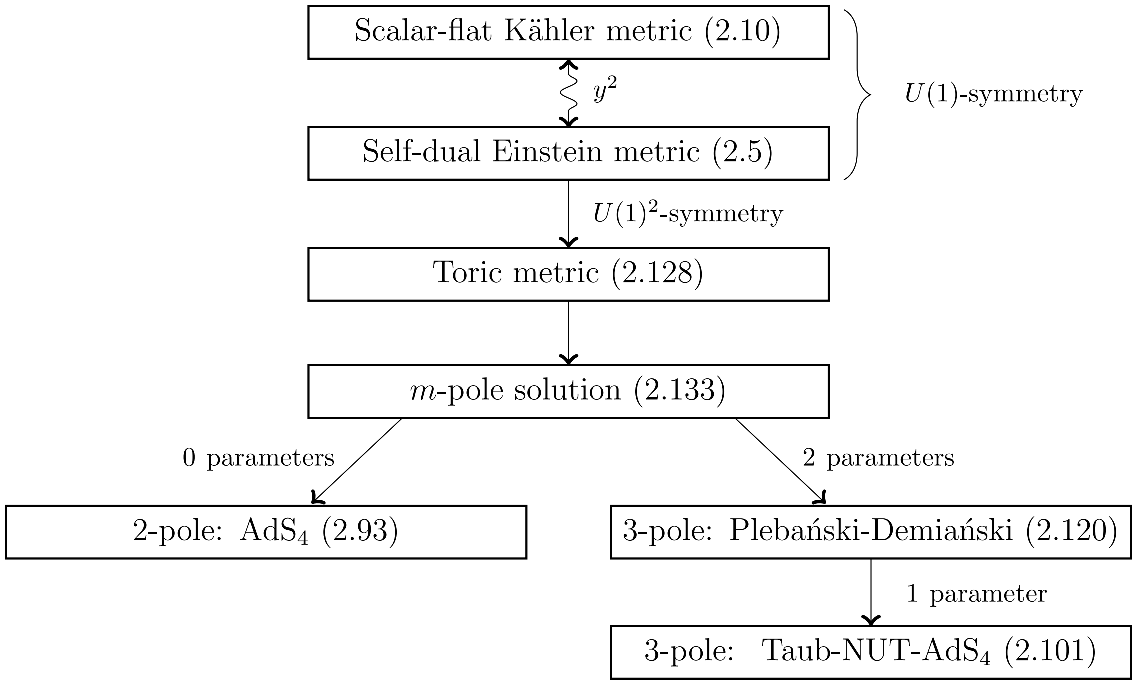

Figure 2.1: Overview of the metrics discussed in this chapter. The arrows point from a metric to a special case of the metric, except the wavy arrow which corresponds to a conformal transformation, *i.e.* equation (2.10).

We have also shown in section 2.4 how AdS_4, Taub-NUT-AdS_4 and the Plebański-Demiański solution fit in this framework. All these previous examples in the literature can be understood as arising from an infinite-dimensional family of local self-dual Einstein metrics with torus symmetry [86]. Figure 2.1 illustrates the relation between all the metrics considered in this chapter. In section 2.4.4 we have suggested that using this family of local metrics, it should be possible to construct global asymptotically locally (Euclidean) AdS self-dual Einstein metrics on the four-ball, thus obtaining an infinite family of completely explicit metrics. It will be interesting to analyze these m-pole solutions in more detail.

In this chapter we have achieved a rather general understanding of the gauge/gravity duality for supersymmetric asymptotically locally Euclidean AdS_4 solutions. Nevertheless, there are a number of possible extensions of our work. First, it is possible to extend the matching of the free energy (2.92) for the class of self-dual backgrounds we have considered to other BPS observables. In particular in [94] the Wilson loop around an orbit of the Killing vector $\mathcal{K}$ was shown to be BPS in the field theory, and may also be computed via localization. The gravity dual is an M2-brane wrapping a calibrated copy of the M-theory circle in the internal space [95], and computing its renormalized action one finds an analogously simple formula to (2.92), namely

$$\lim_{N\to\infty} \log \langle W \rangle \quad = \quad \frac{|b_1| + |b_2|}{2} \ell \cdot \log \langle W \rangle_1 \ , \tag{2.134}$$

where $\langle W \rangle_1$ denotes the large N limit of the Wilson loop on the round sphere/AdS_4, whose log scales as $N^{1/2}$, and $2\pi\ell$ denotes the length of the orbit of $\mathcal{K}$ (for example, such orbits always close over the poles of the S^3, where $\ell = 1/|b_1|$ or $\ell = 1/|b_2|$, respectively; notice that for these Wilson loops (2.134) is again a function only of $|b_1/b_2|$).

One might further generalize our results by relaxing one or more of the assumptions we have made. For example, remaining in the context of minimal gauged supergravity, it would be very interesting to investigate the more general class of supersymmetric, but non-(anti-)self-dual solutions [83]. Several examples of such solutions were constructed in [62, 63], and these all turn out to have a bulk topology different from the four-ball. This suggests that self-duality and the topology of supersymmetric asymptotically AdS$_4$ solutions are two related issues, and it would be desirable to clarify this. On the other hand, at present it is unclear to us what is the precise dual field theory implication of non-trivial two-cycles in the geometry, and therefore this direction is both challenging and interesting. Perhaps related to this, one of our main results is that a smooth toric self-dual Einstein metric on the four-ball with supersymmetric Killing vector $\mathcal{K} = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2}$ gives rise to a smooth supersymmetric solution only if $b_1/b_2 > 0$ or $b_1/b_2 = -1$. Specifically, for other choices of b_1/b_2 the conformal factor and the Killing spinor are singular in the interior of the bulk. Nevertheless, the conformal boundary is smooth for all choices of b_1, b_2, and the question arises as to how to fill those boundaries smoothly within gauged supergravity. A natural conjecture is that these are filled with the non-self-dual solutions mentioned above.

Another assumption that should be straightforward to relax is in taking the gauge field A to be real. In general, if A is complex the existence of one (Euclidean) Killing spinor does not imply that the metric possesses any isometry [83]. However, we expect that if one requires the existence of *two* spinors of opposite R-charge, then there will be canonically defined Killing vectors, and therefore it should be possible to analyze the solutions with the techniques of this chapter.

Chapter 3

Casimir energy of supersymmetric field theories on $\mathbb{R} \times S^3$

In this chapter we study the Casimir energy, that is, the energy of the vacuum state, of $\mathcal{N} = 1$ supersymmetric gauge theories on the cylinder $\mathbb{R} \times S^3$. As discussed in section 1.5, the Casimir energy on the four-dimensional cylinder is in general an ambiguous quantity. However, we will show that for supersymmetric field theories a natural generalization dubbed the *supersymmetric Casimir energy* is well-defined and scheme-independent.

In section 3.1, we set up the background consisting of the round metric on $S^1 \times S^3$ and appropriate background vector fields, and in section 3.2 we discuss the explicit supersymmetric Langrangians. This will all be in Euclidean signature, such that we can study the path integral in section 3.3. We Wick rotate to Lorentzian signature in section 3.4 and study the canonical quantization of these theories on $\mathbb{R} \times S^3$. In section 3.5, by reducing the theory on the S^3 to a one-dimensional quantum mechanics, we show that the supersymmetric Casimir energy is in fact well-defined and scheme-independent if one requires the regularization to be compatible with supersymmetry. Finally, in section 3.6 we make some concluding remarks.

This chapter is based on [2, 3]

3.1 Background geometry

As described in section 1.2.1, a systematic approach for constructing $\mathcal{N} = 1$ supersymmetric field theories with an R-symmetry on curved backgrounds from new minimal supergravity was put forward in [20]. Below, we set up our background consisting of the round metric on $S^1 \times S^3$ as well as the background vector fields.

We consider the following background metric

$$
\begin{aligned}
\mathrm{d}s^2(S^1\times S^3) &= r_1^2\mathrm{d}\tau^2+\mathrm{d}s^2(S^3)\\
&= r_1^2\mathrm{d}\tau^2+\frac{r_3^2}{4}\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2+(\mathrm{d}\varsigma+\cos\theta\mathrm{d}\varphi)^2\right)\ ,
\end{aligned}
\tag{3.1}
$$

where τ is a coordinate on the S^1 with radius r_1, and θ,φ,ς with $0\le\theta<\pi$, $\varphi\sim\varphi+2\pi$, $\varsigma\sim\varsigma+4\pi$ are coordinates on the round three-sphere[1] of radius r_3. We introduce the following orthonormal frame[2]

$$
\begin{aligned}
e^1 &= \frac{r_3}{2}\left(\cos\varsigma\mathrm{d}\theta+\sin\theta\sin\varsigma\mathrm{d}\varphi\right)\\
e^2 &= \frac{r_3}{2}\left(-\sin\varsigma\mathrm{d}\theta+\sin\theta\cos\varsigma\mathrm{d}\varphi\right)\\
e^3 &= \frac{r_3}{2}\left(\mathrm{d}\varsigma+\cos\theta\mathrm{d}\varphi\right)\\
e^4 &= r_1\mathrm{d}\tau\ ,
\end{aligned}
\tag{3.2}
$$

where $\{e^1,e^2,e^3\}$ corresponds to a *left-invariant* frame on S^3. We now set $r_1=1$ and $r_3=2$. We will consider a class of backgrounds admitting a solution to the new minimal Killing spinor equation (1.2). In the coordinates (3.1), the supersymmetric complex Killing vector K reads,

$$
K = \frac{1}{2}\left(\partial_\varsigma-\mathrm{i}\partial_\tau\right)\ , \tag{3.3}
$$

and the dual one-form is

$$
K = \frac{1}{2}\left(e^3-\mathrm{i}e^4\right)\ . \tag{3.4}
$$

We define the following "reference" values of the background fields

$$
\mathring{A} = \frac{3}{4}e^3+\frac{\mathrm{i}}{2}\Big(\mathfrak{q}-\frac{1}{2}\Big)e^4\ ,\qquad \mathring{V} = \frac{1}{2}e^3\ , \tag{3.5}
$$

where we have included a constant $\mathfrak{q}$, which corresponds to a (large) gauge transformation $A\to A+\frac{\mathrm{i}}{2}\mathfrak{q}\mathrm{d}\tau$ starting from the gauge choice adopted in [26]. Although in Euclidean signature, where τ is a compact coordinate, this yields an ill-defined spinor, this is not true in Lorentzian signature, and the $\mathfrak{q}$ will play a role in our discussion in section 3.4. As discussed in [26], the vectors A and V may be shifted from the reference values as

$$
A = \mathring{A}+\frac{3}{2}\kappa K\ ,\qquad V = \mathring{V}+\kappa K\ . \tag{3.6}
$$

[1]For $r_1=1$ and $r_3=2$ this metric and the other background fields can be obtained by specializing the background discussed in appendix C of [26] to $v=1$, $b_1=-b_2=1/2$. Below we will set $r_1=1$ and $r_3=2$, but these can be easily restored by dimensional analysis.

[2]Note that this frame is *different* from the frame used in [26].

where κ is a constant[3]. We note the combination

$$A^{\rm cs} \;\equiv\; A - \frac{3}{2}V \;=\; \mathring{A} - \frac{3}{2}\mathring{V} \;=\; \frac{\mathrm{i}}{2}\big(\mathfrak{q} - \frac{1}{2}\big)e^4 \;, \tag{3.7}$$

which is independent of κ. For generic values of κ, the two-component spinor ζ solving the Killing spinor equation (1.2) reads

$$\zeta \;=\; \frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{1}{2}\mathfrak{q}\tau}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;. \tag{3.8}$$

The normalization is chosen such that for $\mathfrak{q} = 0$ the square norm is $|\zeta|^2 = 1/2$ as in [26]. As mentioned above, since τ is periodic in Euclidean signature, the spinor is well-defined only[4] for $\mathfrak{q} = 0$.

For generic values of κ this background preserves only an $SU(2)_l \times U(1)_r$ subgroup of the isometry group $SO(4) \simeq SU(2)_l \times SU(2)_r$ of the round three-sphere. Two choices for κ will be of special interest below. The value $\kappa = \kappa^{\rm ACM} = -1/3$ corresponds (for $\mathfrak{q} = 0$) to that in [26], namely

$$A^{\rm ACM} = \frac{1}{2}e^3 \;, \qquad V^{\rm ACM} \;=\; \frac{1}{3}\left(e^3 + \frac{\mathrm{i}}{2}e^4\right) \;. \tag{3.9}$$

This is the particular choice of κ (for $\mathfrak{q} = 0$) for which A is real. Another distinguished choice is $\kappa = \kappa^{\rm st} = -1$, where the superscript stands for "standard", giving

$$A^{\rm st} \;=\; \frac{\mathrm{i}}{2}\left(1 + \mathfrak{q}\right)e^4 \;, \qquad V^{\rm st} \;=\; \frac{\mathrm{i}}{2}e^4 \;. \tag{3.10}$$

For this choice, the full $SO(4)$ symmetry of the three-sphere is restored and equation (1.2) admits a more general solution,

$$\zeta \;=\; \mathrm{e}^{-\frac{1}{2}\mathfrak{q}\tau}\zeta_0 \;, \tag{3.11}$$

for any constant spinor ζ_0.

Notice that in addition to $\mathfrak{q} = 0$ [26], there are two special values of the parameter $\mathfrak{q}$. Namely, $\mathfrak{q} = -1$ for which $A^{\rm st}$ in (3.10) vanishes, and $\mathfrak{q} = 1/2$ for which $A^{\rm cs} = 0$ from equation (3.7). The significance of these three values will become clearer in the later sections.

[3] In general, κ can be a complex function satisfying $K^\mu\partial_\mu\kappa = 0$.

[4] Although an appropriately quantized imaginary value of $\mathfrak{q}$ would be allowed in (3.8), for generic R-charges we must have $\mathfrak{q} = 0$ for the correct periodicity of the matter fields [96].

3.2 Lagrangians

We consider an $\mathcal{N}=1$ supersymmetric field theory with a vector multiplet transforming in the adjoint representation of the gauge group G, and a chiral multiplet transforming in a representation $\mathcal{R}$. We will restrict attention to terms in the Lagrangian quadratic in dynamical fields, as we are interested in the energy of the vacuum.

For a chiral multiplet with R-charge r, we consider the Lagrangian

$$\begin{aligned}\mathcal{L}^{\text{chiral}} &= (\delta_\zeta V_1+\delta_\zeta V_2+\epsilon\,\delta_\zeta V_U)\,|_{\text{quadratic}}\\ &= D_\mu\widetilde{\phi}D^\mu\phi+(V^\mu+(\epsilon-1)U^\mu)\left(\mathrm{i}D_\mu\widetilde{\phi}\phi-\mathrm{i}\widetilde{\phi}D_\mu\phi\right)+\frac{r}{4}(R+6V_\mu V^\mu)\widetilde{\phi}\phi\\ &\quad +\mathrm{i}\widetilde{\psi}\widetilde{\sigma}^\mu D_\mu\psi+\left(\frac{1}{2}V^\mu+(1-\epsilon)U^\mu\right)\widetilde{\psi}\widetilde{\sigma}_\mu\psi\ ,\end{aligned}\tag{3.12}$$

with $D_\mu=\nabla_\mu-\mathrm{i}q_RA_\mu$ where q_R denotes the R-charges of the fields. The three terms in first line are total supersymmetry variations and given explicitly in [26][5]. We included a parameter ϵ, such that (3.12) can continuously interpolate between the localizing Lagrangian used in [26] with $\epsilon=0$, and the usual chiral multiplet [25] obtained for $\epsilon=1$.

The vector fields A_μ, V_μ and $U_\mu=\kappa K_\mu$ are those discussed in section 3.1, depending on the parameters $\mathfrak{q}$ and κ. A Lagrangian containing $N_\chi=|\mathcal{R}|$ chiral multiplets consists of just multiples of (3.12), and each multiplet may have different R-charge r_I, where $I=1,2,...,N_\chi$.

We employ the left-invariant frame (3.2), which is useful for applying the angular momentum formalism. The 2×2 sigma matrices,

$$\sigma^A_{\alpha\dot{\alpha}} \;=\; (\vec{\gamma},-\mathrm{i}\mathbb{1}_2)\ ,\qquad \widetilde{\sigma}^A_{\alpha\dot{\alpha}} \;=\; (-\vec{\gamma},-\mathrm{i}\mathbb{1}_2)\ ,\tag{3.13}$$

where $A=1,...,4$ is a frame index and $\vec{\gamma}$ denotes the three Pauli matrices, generate the Euclidean Clifford algebra,

$$\sigma_A\widetilde{\sigma}_B+\sigma_B\widetilde{\sigma}_A \;=\; -2\delta_{AB}\ ,\qquad \widetilde{\sigma}_A\sigma_B+\widetilde{\sigma}_B\sigma_A \;=\; -2\delta_{AB}\ .\tag{3.14}$$

Inserting the values of the background fields, and writing

$$\mathcal{L}^{\text{chiral}}(\mathfrak{q},\kappa,\epsilon,r) \;=\; \mathcal{L}^{\text{chiral}}_{\text{bos}}(\mathfrak{q},\kappa,\epsilon,r)+\mathcal{L}^{\text{chiral}}_{\text{fer}}(\mathfrak{q},\kappa,\epsilon,r)\tag{3.15}$$

[5]Notice that at quadratic order, the term $\delta_\zeta V_3$ in [26] vanishes.

the bosonic part of the Lagrangian reads

$$
\begin{aligned}
\mathcal{L}^{\text{chiral}}_{\text{bos}}(\mathfrak{q},\kappa,\epsilon,r) \;=\;& -\widetilde{\phi}\partial_\tau^2\phi + \left[\frac{r}{2}(1-2\mathfrak{q}) + \kappa\left(\frac{3}{2}r-\epsilon\right)\right]\widetilde{\phi}\partial_\tau\phi - \widetilde{\phi}\nabla^i\nabla_i\phi \\
& + \mathrm{i}\left[\frac{3}{2}r - 1 + \kappa\left(\frac{3}{2}r-\epsilon\right)\right]\widetilde{\phi}\nabla_\varsigma\phi \\
& + \frac{r}{2}(1+\mathfrak{q})\left[\frac{r}{2}(2-\mathfrak{q}) + \kappa\left(\frac{3}{2}r-\epsilon\right)\right]\widetilde{\phi}\phi \;, \qquad (3.16)
\end{aligned}
$$

where ∇_i is the covariant derivative on the three-sphere, and we have omitted a total derivative. The fermionic part of the Lagrangian reads

$$
\begin{aligned}
\mathcal{L}^{\text{chiral}}_{\text{fer}}(\mathfrak{q},\kappa,\epsilon,r) \;=\;& \widetilde{\psi}\partial_\tau\psi - \mathrm{i}\widetilde{\psi}\gamma^a\partial_a\psi - \frac{1}{2}\left[\frac{3}{2}r - 1 + \kappa\left(\frac{3}{2}r-\epsilon\right)\right]\widetilde{\psi}\gamma_\varsigma\psi \\
& - \frac{1}{2}\left[\frac{1}{2}(r-1)(1-2\mathfrak{q}) + \frac{3}{2} + \kappa\left(\frac{3}{2}r-\epsilon\right)\right]\widetilde{\psi}\psi \;, \qquad (3.17)
\end{aligned}
$$

where $a = 1,2,3$ denotes the frame index on the three-sphere. In particular, we used the identity

$$
\mathrm{i}\widetilde{\sigma}^\mu\nabla_\mu\psi \;=\; \partial_\tau\psi - \mathrm{i}\gamma^a\nabla_a\psi \;=\; \partial_\tau\psi - \mathrm{i}\gamma^a\partial_a\psi - \frac{3}{4}\psi \;. \qquad (3.18)
$$

Notice that the Lagrangians in [55, 57] correspond to the values $\epsilon = 1$, $\kappa = -1$, and $\mathfrak{q} = 1/2$. Notice also that for $r = 2/3$ and $\epsilon = 1$ the total chiral multiplet Lagrangian does not depend on κ.

Let us introduce a compact notation, writing the Lagrangians above in terms of differential operators. Denoting by ℓ_a the Killing vectors dual to the left-invariant frame e^a, and defining the "orbital" angular momentum operators as $L_a = \frac{\mathrm{i}}{2}\ell_a$, one finds that these satisfy the $SU(2)$ commutation relations,

$$
[L_a, L_b] \;=\; \mathrm{i}\epsilon_{abc}L_c \;, \qquad (3.19)
$$

and we have[6] $-\nabla^i\nabla_i = \vec{L}^2$ and $\nabla_\varsigma = -\mathrm{i}L_3$. Similarly, we identity the Pauli matrices with the spin operator as $S^a = \frac{1}{2}\gamma^a$, satisfying the same $SU(2)$ algebra. The Lagrangians can then be writing as

$$
\begin{aligned}
\mathcal{L}^{\text{chiral}}_{\text{bos}} \;&=\; \widetilde{\phi}\widetilde{\mathcal{O}}_b\phi \;=\; \widetilde{\phi}\left(-\partial_\tau^2 + 2\mu\partial_\tau + \mathcal{O}_b\right)\phi \;, \\
\mathcal{L}^{\text{chiral}}_{\text{fer}} \;&=\; \widetilde{\psi}\widetilde{\mathcal{O}}_f\psi \;=\; \widetilde{\psi}\left(\partial_\tau + \mathcal{O}_f\right)\psi \;, \qquad (3.20)
\end{aligned}
$$

[6]Recall that here we have set $r_3 = 2$. In general, the three-dimensional Laplace operator is $r_3^2\nabla^i\nabla_i = \sum_a(\ell_a)^2$.

where

$$\begin{aligned} \mathcal{O}_b &= 2\alpha_b \vec{L}^2 + 2\beta_b L_3 + \gamma_b \ , \\ \mathcal{O}_f &= 2\alpha_f \vec{L}\cdot\vec{S} + 2\beta_f S_3 + \gamma_f \ , \end{aligned} \tag{3.21}$$

with the constants taking the values $\alpha_b = \frac{1}{2}$,

$$\begin{aligned} \beta_b &= -\frac{1}{2} + \frac{3}{4}r + \frac{\kappa}{2}\left(\frac{3}{2}r - \epsilon\right) \ , \\ \gamma_b &= \frac{r}{2}(1+\mathfrak{q})\left[\frac{r}{2}(2-\mathfrak{q}) + \kappa\left(\frac{3}{2}r - \epsilon\right)\right] \ , \\ \mu &= \frac{1}{2}\left[\frac{r}{2}(1-2\mathfrak{q}) + \kappa\left(\frac{3}{2}r - \epsilon\right)\right] \ , \end{aligned} \tag{3.22}$$

and $\alpha_f = -1$, $\beta_f = -\beta_b$,

$$\gamma_f = -\left[\frac{1}{4}(r-1)(1-2\mathfrak{q}) + \frac{3}{4} + \frac{\kappa}{2}\left(\frac{3}{2}r - \epsilon\right)\right] \ , \tag{3.23}$$

respectively.[7]

For the vector multiplet, the quadratic Lagrangian is

$$\mathcal{L}^{\text{vector}} = \text{Tr}\left[\frac{1}{4}\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu} + \frac{\mathrm{i}}{2}\lambda\sigma^\mu D^{\text{cs}}_\mu\widetilde{\lambda} + \frac{\mathrm{i}}{2}\widetilde{\lambda}\widetilde{\sigma}^\mu D^{\text{cs}}_\mu\lambda\right]_{\text{quadratic}} \ , \tag{3.24}$$

where $D^{\text{cs}}_\mu = \nabla_\mu - \mathrm{i}q_R A^{\text{cs}}_\mu$. At quadratic order, $\mathsf{F}_{\mu\nu}$ is the linearized field strength, $\mathsf{F} = \mathrm{d}\mathsf{A}$, of the gauge field A. The gaugino λ and A both transform in the adjoint representation of the gauge group G. This is therefore the Lagrangian of $N_v = |G|$ free vector multiplets, where we denote by N_v the dimension of G.

The fermionic part of this Lagrangian can be put in the same form as the fermionic part of the chiral multiplet Lagrangian, namely

$$\mathcal{L}^{\text{vector}}_{\text{fer}} = \widetilde{\lambda}\widetilde{\mathcal{O}}^{\text{vec}}_f\lambda = \widetilde{\lambda}\left(\partial_\tau + \mathcal{O}^{\text{vec}}_f\right)\lambda \ , \tag{3.25}$$

where

$$\mathcal{O}^{\text{vec}}_f = 2\alpha_v\vec{L}\cdot\vec{S} + 2\beta_v S_3 + \gamma_v \ , \tag{3.26}$$

with $\alpha_v = -1$, $\beta_v = 0$, and $\gamma_v = \frac{\mathfrak{q}}{2} - 1$. Notice that for $\mathfrak{q} = 1/2$, corresponding to $A^{\text{cs}} = 0$, this reduces to the standard massless Dirac operator on the three-sphere.

[7]Recall we denote the Pauli matrices as γ_a, while we defined here parameters γ_b and γ_f. Hopefully, this will not lead to confusion.

3.3 Path integral approach

In this section, we recover in our set-up the supersymmetric Casimir energy defined in [26] as

$$E_{\text{susy}} \;=\; -\lim_{\beta\to\infty}\frac{\mathrm{d}}{\mathrm{d}\beta}\log Z(\beta)\ , \tag{3.27}$$

where Z is the *supersymmetric partition function*, namely the path integral on $S^1\times S^3$ with periodic boundary conditions for the fermions on S^1, computed using localization. Restoring the radii of S^1 and S^3, the dimensionless parameter β of [26] is given by

$$\beta \;=\; \frac{2\pi r_1}{r_3}\ . \tag{3.28}$$

Differently from [26], here we will not fix the value of κ, showing that owing to the pairing of bosonic and fermionic eigenvalues in the one-loop determinant, the final result will be *independent* of κ. Although the computation in Euclidean signature requires to fix $\mathfrak{q}=0$, we start presenting the explicit eigenvalues for generic values of $\mathfrak{q}$. We will demonstrate that the pairing occurs if and only if $\mathfrak{q}=0$.

The localization computation of [26] shows that the partition function takes the form

$$Z(\beta) \;=\; \mathrm{e}^{-\mathcal{F}(\beta)}\mathcal{I}(\beta)\ , \tag{3.29}$$

where $\mathcal{I}(\beta)$ is the supersymmetric index [55–58], and the pre-factor $\mathcal{F}(\beta)=-\mathrm{i}\pi\big(\Psi^{(0)}_{\text{chi}}+\Psi^{(0)}_{\text{vec}}\big)$ arises from the regularization of one-loop determinants in the chiral multiplets and vector multiplets, respectively ([26], see also [97]). The index $\mathcal{I}(\beta)$ does not contribute in the limit (3.27), so we can focus on $\Psi^{(0)}_{\text{chi}}$ and $\Psi^{(0)}_{\text{vec}}$, and thus effectively set the constant gauge field $\mathcal{A}_0=0$ in the one-loop determinants around the localization locus in [26]. Since the vector multiplet (3.24) does not depend on κ and ϵ, and setting $\mathfrak{q}=0$, its contribution to E_{susy} can simply be borrowed from [26]. Setting $|b_1|=|b_2|=\beta/(2\pi)=r_1/r_3$, where we used (3.28), one obtains

$$\Psi^{(0)}_{\text{vec}} \;=\; \frac{\mathrm{i}}{6}\left(\frac{r_1}{r_3}-\frac{r_3}{r_1}\right)N_v\ . \tag{3.30}$$

For the chiral multiplet, we first work out the eigenvalues of the operators $\mathcal{O}_b$ and $\mathcal{O}_f$ for arbitrary κ and ϵ. The eigenvalues can be obtained with elementary methods from the theory of angular momentum in quantum mechanics [44]. See appendix A for a summary of the relevant spherical harmonics on the three-sphere. We denote the eigenvalues as

$$\mathcal{O}_b\phi \;=\; E_b^2\phi\ ,\qquad \mathcal{O}_f\psi \;=\; \lambda^{\pm}\psi\ . \tag{3.31}$$

From the scalar harmonics, we have the eigenvalues

$$E_b^2 \;=\; \frac{\alpha_b}{2}\ell(\ell+2)+2\beta_b m+\gamma_b\ , \tag{3.32}$$

where $\frac{\ell}{2}(\frac{\ell}{2}+1)$ for $\ell=0,1,2,...$ are the eigenvalues of $\vec{L}^2$, and $m=-\frac{\ell}{2},...,\frac{\ell}{2}$ are the eigenvalues of L_3. Each eigenvalue has degeneracy $(\ell+1)$, due to the $SU(2)_r$ symmetry.

We distinguish two types of eigenvalues of $\mathcal{O}_f$. For any $\ell=1,2,3,..$ we have

$$\lambda^{\pm}_{\ell m} \;=\; -\frac{\alpha_f}{2}+\gamma_f \pm \sqrt{\frac{\alpha_f^2}{4}(\ell+1)^2+\alpha_f\beta_f(1+2m)+\beta_f^2}\ , \tag{3.33}$$

where here the quantum number m takes the values $m=-\frac{\ell}{2},...,\frac{\ell}{2}-1$. Furthermore, for any $\ell=0,1,2,...$, we have the two special eigenvalues

$$\lambda^{\text{special}\pm}_{\ell} \;=\; \frac{\alpha_f}{2}\ell \pm \beta_f+\gamma_f\ . \tag{3.34}$$

Again, each eigenvalue has degeneracy $(\ell+1)$, due to the $SU(2)_r$ symmetry. Expanding the fields in Kaluza-Klein modes on the S^1 as

$$\phi(x) \;=\; \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}k\tau}\phi_k(\theta,\varphi,\varsigma)\ , \tag{3.35}$$

and similarly for ψ, we obtain the following eigenvalues for each mode,

$$\begin{aligned} \widetilde{\mathcal{O}}_b\phi_k \;&=\; \left(k^2-2\mathrm{i}\mu k+E_b^2\right)\phi_k\ ,\\ \widetilde{\mathcal{O}}_f\psi_k \;&=\; \left(-\mathrm{i}k+\lambda^{\pm}\right)\psi_k\ . \end{aligned} \tag{3.36}$$

For generic values of the quantum numbers ℓ,m, we say that the eigenvalues of the operators $\widetilde{\mathcal{O}}_b$ and $\widetilde{\mathcal{O}}_f$ are *paired*, if for all k we have

$$\left(-\mathrm{i}k+\lambda^+\right)\left(-\mathrm{i}k+\lambda^-\right) \;=\; -\left(k^2-2\mathrm{i}\mu k+E_b^2\right)\ . \tag{3.37}$$

Inserting the values of the parameters given in (3.22) and (3.23) we find that the eigenvalues pair if and only if $\mathfrak{q}=0$, in which case they pair for any κ,ϵ,r. Let us set $\mathfrak{q}=0$ in the rest of this section. Restoring generic values of the radius r_3 of the S^3, the one-loop determinant for a fixed k is

$$Z^{(k)}_{1-\text{loop}} \;=\; \frac{\det\widetilde{\mathcal{O}}_f}{\det\widetilde{\mathcal{O}}_b} \;=\; \frac{\prod_{\lambda^-}\left(-\mathrm{i}k+\frac{2}{r_3}\lambda^-\right)\prod_{\lambda^+}\left(-\mathrm{i}k+\frac{2}{r_3}\lambda^+\right)}{\prod_{E_b}\left(k^2-\frac{4}{r_3}\mathrm{i}\mu k+\frac{4}{r_3^2}E_b^2\right)}\ , \tag{3.38}$$

where the products are over all the bosonic and fermionic eigenvalues, including the special ones. However, using the condition (3.37) all the paired eigenvalues cancel out.[8] For $m=\ell/2$ the generic fermionic eigenvalues do not exist. Thus there are *unpaired* bosonic eigenvalues remaining the denominator of (3.38). These are

[8] Up to an irrelevant sign.

obtained setting $m = \ell/2$ in (3.32), which reads

$$(E_b^2)^{\text{unpaired}} \;=\; \left(\frac{\alpha_f}{2}(\ell+1)+\beta_f\right)^2 - \mu^2 \;, \qquad \ell \;=\; 0,1,2,\ldots \quad . \tag{3.39}$$

In the numerator of (3.38) remain the special fermionic eigenvalues. Thus the one-loop determinant for fixed k, including the degeneracies, reads,

$$Z^{(k)}_{1-\text{loop}} \;=\; \prod_{n_0=1}^{\infty}\left(\frac{n_0+1+r_3\mathrm{i}k-r}{n_0-1-r_3\mathrm{i}k+r}\right)^{n_0} \;, \tag{3.40}$$

where we defined $n_0 = \ell+1$ and used that $\alpha_f = -1$ and

$$\beta_f + \mu \;=\; \frac{1}{2}(1-r) \;. \tag{3.41}$$

Upon obvious identifications, this coincides with the one-loop determinant of an $\mathcal{N}=2$ chiral multiplet on the round three-sphere, originally derived in [47] and [48], although our operators $\mathcal{O}_b$ and $\mathcal{O}_f$ are slightly more general and interpolate between those used in these references. In particular, the Lagrangian used in [47] corresponds to $\kappa = -1/3$ and $\epsilon = 0$, precisely as in [26], while those used in [48] correspond to $\kappa = -1$ and $\epsilon = 1$. Recall that in all cases we have set $\mathfrak{q} = 0$.

Defining

$$z \;=\; 1 - r + \frac{r_3\mathrm{i}k}{r_1} \;, \tag{3.42}$$

where we restored the radius r_1 of the S^1, one finds $Z^{(k)}_{1-\text{loop}}(z) \;=\; s_{b=1}(\mathrm{i}z)$, where $s_b(x)$ is the double sine function (1.24). Alternatively, (3.40) can be written in terms of special functions by integrating the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\log Z^{(k)}_{1-\text{loop}} \;=\; -\pi z\cot(\pi z) \;, \tag{3.43}$$

where the Hurwitz function has been used to regularize the infinite sum [47] (see appendix B).

In order to take the limit $\beta \to \infty$, it is more convenient to write (3.40) as an infinite product over two integers, namely

$$Z^{(k)}_{1-\text{loop}} \;=\; \prod_{n_1=0}^{\infty}\prod_{n_2=0}^{\infty}\frac{n_1+n_2+1+z}{n_1+n_2+1-z} \;. \tag{3.44}$$

The full partition function is obtained as as product over the Kaluza-Klein modes,

$$Z_{1-\text{loop}} \;=\; \prod_{k\in\mathbb{Z}} Z^{(k)}_{1-\text{loop}} \;. \tag{3.45}$$

Following the regularization in [26], we write the one-loop determinant in terms of triple gamma functions,

$$
\begin{aligned}
Z_{1-\text{loop}} &= \prod_{k,n_1,n_2=0}^{\infty} \frac{k+u(n_1+n_2+2-r)}{k+u(n_1+n_2+r)} \cdot \frac{k+1-u(n_1+n_2+2-r)}{k+1-u(n_1+n_2+r)} \\
&= \frac{u_3\big(ur|1,u,u\big)u_3\big(1-ur|1,-u,-u\big)}{u_3\big(u(2-r)|1,u,u\big)u_3\big(1-u(2-r)|1,-u,-u\big)} \ , \qquad (3.46)
\end{aligned}
$$

where $u = \mathrm{i}r_1/r_3$. From equations (6.4) and (5.24) in [98], this leads to

$$
Z_{1-\text{loop}} = \mathrm{e}^{\mathrm{i}\pi\Psi^{(0)}_{\text{chi}}}\widetilde{\Gamma}_e\left(\frac{\mathrm{i}r_1}{r_3}r, \frac{\mathrm{i}r_1}{r_3}, \frac{\mathrm{i}r_1}{r_3}\right) \ , \qquad (3.47)
$$

with

$$
\Psi^{(0)}_{\text{chi}} = \frac{\mathrm{i}}{6}\left[\frac{2r_1}{r_3}(r-1)^3 - \left(\frac{r_3}{r_1}+\frac{r_1}{r_3}\right)(r-1)\right] \ , \qquad (3.48)
$$

where $\widetilde{\Gamma}_e(w,p,q) = \Gamma_e\big(\mathrm{e}^{2\pi\mathrm{i}w}, \mathrm{e}^{2\pi\mathrm{i}p}, \mathrm{e}^{2\pi\mathrm{i}q}\big)$ is the elliptic gamma function,

$$
\Gamma_e(w,p,q) = \prod_{m,n\geq 0} \frac{1-p^{m+1}q^{n+1}w^{-1}}{1-p^m q^n w} \ . \qquad (3.49)
$$

From this, one finds the contribution of a chiral multiplet to (3.27) to be

$$
E^{\text{chiral}}_{\text{susy}} = \frac{1}{12}\left(2(r-1)^3 - (r-1)\right) \ . \qquad (3.50)
$$

This is exactly the contribution of a chiral multiplet with R-charge r to the total supersymmetric Casimir energy computed in [26], although we emphasize that here this has been derived for arbitrary values of the parameters κ and ϵ.

Combining the contributions from the vector multiplets (3.30) and the chiral multiplets we recover the result

$$
E_{\text{susy}} = \frac{4}{27}\left(\mathbf{a}+3\mathbf{c}\right) \ , \qquad (3.51)
$$

with the anomaly coefficients defined as

$$
\mathbf{a} = \frac{3}{32}\left(3\mathrm{tr}\mathbf{R}^3 - \mathrm{tr}\mathbf{R}\right) \ , \qquad \mathbf{c} = \frac{1}{32}\left(9\mathrm{tr}\mathbf{R}^3 - 5\mathrm{tr}\mathbf{R}\right) \ , \qquad (3.52)
$$

where $\mathbf{R}$ denotes the R-symmetry charge, and "tr" runs over the fermionic fields in the multiplets, so that for N_v vector multiplets and N_χ chiral multiplets,

$$
\mathrm{tr}\mathbf{R}^n = N_v + \sum_{I=1}^{N_\chi}(r_I-1)^n \ . \qquad (3.53)
$$

In the next section we will show that (3.51) is also equal to the expectation value of the BPS Hamiltonian H_{susy} appearing in the supersymmetric index

$$\mathcal{I}(\beta) \;=\; \mathrm{Tr}\,(-1)^F \mathrm{e}^{-\beta H_{\text{susy}}} \;. \tag{3.54}$$

Therefore we now turn to the Hamiltonian formalism, working in a background with a non-compact time direction, thus with $\beta \to \infty$ from the outset.

3.4 Hamiltonian formulation

In this section we study the theories defined in section 3.2 in a background $\mathbb{R}\times S^3$ in Lorentzian signature, obtained from the geometry in section 3.1 by a simple analytic continuation. In particular, we take the metric

$$\mathrm{d}s^2(\mathbb{R}\times S^3) \;=\; -\mathrm{d}t^2 + \mathrm{d}s^2(S^3) \;, \tag{3.55}$$

where t denotes the time coordinate on $\mathbb{R}$, and $\mathrm{d}s^2(S^3)$ is the metric on the round S^3, equation (3.1). Below we continue to set $r_3 = 2$. The background fields are obtained from those of the previous section by setting $A_t = -\mathrm{i}A_\tau$, $V_\tau = -\mathrm{i}V_\tau$, and $K_t = -\mathrm{i}K_\tau$. We must have $\kappa \in \mathbb{R}$, such that the background fields are real. Moreover, the dynamical fields in Lorentzian signature obey $\widetilde{\phi} = \phi^\dagger$ and $\widetilde{\psi} = \psi^\dagger$. The σ-matrices generating the appropriate Clifford algebra are obtained setting $\sigma^0_{\alpha\dot{\alpha}} = \mathrm{i}\sigma^4_{\alpha\dot{\alpha}} = \mathbb{1}_{\alpha\dot{\alpha}}$ and $\widetilde{\sigma}^0_{\alpha\dot{\alpha}} = \mathrm{i}\widetilde{\sigma}^4_{\alpha\dot{\alpha}} = \mathbb{1}_{\alpha\dot{\alpha}}$, with the remaining components unchanged, such that

$$\sigma_A\widetilde{\sigma}_B + \sigma_B\widetilde{\sigma}_A \;=\; -2\eta_{AB} \;, \qquad \widetilde{\sigma}_A\sigma_B + \widetilde{\sigma}_B\sigma_A \;=\; -2\eta_{AB} \;, \tag{3.56}$$

where now $A = 0, ..., 3$ and $\eta_{AB} = \mathrm{diag}(-1,1,1,1)$. The Lorentzian spinor ζ solving equation (1.2) for generic κ is then

$$\zeta \;=\; \frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}qt}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;, \tag{3.57}$$

again with a more general solution for the special value $\kappa = \kappa^{\text{st}} = -1$ [55, 99].

3.4.1 Conserved charges

In the following we consider a chiral multiplet and we will drop the superscript "chiral" from all quantities. The Hamiltonian density $\mathcal{H} = \mathcal{H}_{\text{bos}} + \mathcal{H}_{\text{fer}}$, associated to the chiral multiplet Lagrangian (3.12), is obtained as usual by defining the canonical momenta

$$\Pi \;=\; \partial_t\widetilde{\phi} - \mathrm{i}\mu\widetilde{\phi} \;, \qquad \widetilde{\Pi} \;=\; \partial_t\phi + \mathrm{i}\mu\phi \;, \qquad \pi^\alpha \;=\; \mathrm{i}\widetilde{\psi}_{\dot{\alpha}}\widetilde{\sigma}^{0\,\dot{\alpha}\alpha} \;, \qquad \widetilde{\pi}^\alpha \;=\; 0 \;, \tag{3.58}$$

and its bosonic and fermionic parts read

$$\begin{aligned}\mathcal{H}_{\text{bos}} &= \Pi\partial_t\phi + \widetilde{\Pi}\partial_t\widetilde{\phi} - \mathcal{L}_{\text{bos}}^{\text{chiral}} ,\\ \mathcal{H}_{\text{fer}} &= \pi\partial_t\psi + \widetilde{\pi}\partial_t\widetilde{\psi} - \mathcal{L}_{\text{fer}}^{\text{chiral}} ,\end{aligned} \tag{3.59}$$

respectively. In terms of the operators $\mathcal{O}_b$ and $\mathcal{O}_f$ defined in equations (3.21), we have

$$\begin{aligned}\mathcal{H}_{\text{bos}} &= \widetilde{\Pi}\Pi - \mathrm{i}\mu\big(\Pi\phi - \widetilde{\Pi}\widetilde{\phi}\big) + \widetilde{\phi}\left(\mathcal{O}_b + \mu^2\right)\phi ,\\ \mathcal{H}_{\text{fer}} &= -\widetilde{\psi}\mathcal{O}_f\psi .\end{aligned} \tag{3.60}$$

The Hamiltonian is then obtained by integrating[9] over the spatial S^3,

$$H = \int \sqrt{g_3}\mathrm{d}^3x\ \mathcal{H} . \tag{3.61}$$

The R-symmetry current J_{R}^{μ} can be derived either from the Noether procedure or as the functional derivative of the action with respect to A_μ, namely

$$J_{\mathrm{R}}^{\mu} = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta A_\mu} , \tag{3.62}$$

and it reads

$$J_{\mathrm{R}}^{\mu} = \mathrm{i}r\big(D^\mu\widetilde{\phi}\,\phi - \widetilde{\phi}D^\mu\phi\big) + 2r\big(V^\mu + \kappa(\epsilon-1)K^\mu\big)\widetilde{\phi}\phi + (r-1)\widetilde{\psi}\widetilde{\sigma}^\mu\psi . \tag{3.63}$$

This is conserved, *i.e.* $\nabla_\mu J_{\mathrm{R}}^{\mu} = 0$, and the corresponding conserved charge R is obtained by contracting it with the time-like Killing vector ∂_t, and integrating on the S^3, which yields

$$R = \int \sqrt{g_3}\mathrm{d}^3x\left(\mathrm{i}r\big(\widetilde{\phi}\widetilde{\Pi} - \phi\Pi\big) + (r-1)\widetilde{\psi}\widetilde{\sigma}^t\psi\right) . \tag{3.64}$$

Rotational symmetry along the Killing vector ∂_ς, which belongs to the $SU(2)_l$ part of the isometry group of the sphere, gives rise to a conserved current with the corresponding conserved angular momentum

$$J_3 = -\mathrm{i}\int \sqrt{g_3}\mathrm{d}^3x\left(\left(L_3\phi\right)\Pi + (L_3\widetilde{\phi})\,\widetilde{\Pi} + \mathrm{i}\widetilde{\psi}\,\left(L_3 + S_3\right)\psi\right) . \tag{3.65}$$

Finally, supersymmetry gives rise to the conserved supercurrent

$$\zeta^\alpha J^\mu_{\text{susy}\ \alpha} = -\sqrt{2}\zeta\sigma^\nu\widetilde{\sigma}^\mu\psi D_\nu\widetilde{\phi} . \tag{3.66}$$

[9]The integral is over the spatial S^3 with the metric $\mathrm{d}s^2(S^3)$ in (3.1). We define $\mathrm{d}^3x = \mathrm{d}\theta\mathrm{d}\varsigma\mathrm{d}\varphi$, and $g_3 = \sin^2\theta$ denotes the determinant of this metric.

Using the equations of motion for the dynamical fields, after some calculations, one can verify that

$$\nabla_\mu(\zeta J^\mu_{\rm susy}) \;=\; 0 \;. \tag{3.67}$$

Note that $\nabla_\mu\zeta \neq 0$, and therefore $J^\mu_{\rm susy}$ is not conserved by itself, as is the case in the standard flat-space computation. Contracting $\zeta J^\mu_{\rm susy}$ with the time-like Killing vector ∂_t, we obtain the conserved supercharge

$$\mathcal{Q} \;=\; -\sqrt{2}\int \mathrm{d}^3x\sqrt{g_3}\left(\zeta\psi\,\Pi - \mathrm{i}\widetilde{\phi}\zeta\widehat{\mathcal{O}}_f\psi\right) \;, \tag{3.68}$$

where we defined

$$\widehat{\mathcal{O}}_f \;\equiv\; 2\widehat{\alpha}\vec{S}\cdot\vec{L} + 2\widehat{\beta}S_3 + \widehat{\gamma} \;, \tag{3.69}$$

with

$$\widehat{\alpha} \;=\; -1 \;, \qquad \widehat{\beta} \;=\; \frac{3}{4}(1-r) \;, \qquad \widehat{\gamma} \;=\; -\frac{\kappa}{2}\left(\frac{3}{2}r - \epsilon\right) - \frac{3}{4} \;. \tag{3.70}$$

In summary, applying the Noether procedure to the Lagrangian (3.12), we have derived expressions for the Hamiltonian H, R-charge R, angular momentum J_3, and supercharge $\mathcal{Q}$. These will provide the relevant operators in the quantized theory.

Let us briefly discuss other currents that can be considered, which however are not conserved generically. In particular, the usual energy-momentum tensor, defined as

$$T_{\mu\nu} \;=\; \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \;, \tag{3.71}$$

is *not conserved* in the presence of non-dynamical fields. This remains true even if $T_{\mu\nu}$ is contracted with a vector field that generates a symmetry of the metric and the other background fields. Thus, for example, T_{tt} does not define a conserved quantity, and in particular it does not coincide with the canonical Hamiltonian. Denoting generic non-dynamical vector fields as A^I, with $F^I = \mathrm{d}A^I$, and the associated currents as J^μ_I, in general the energy-momentum tensor (3.71) obeys the Ward identity

$$\nabla^\mu T_{\mu\nu} \;=\; \sum_I \left(F^I_{\mu\nu}J^\mu_I - A^I_\nu\nabla_\mu J^\mu_I\right) \;. \tag{3.72}$$

In the present case, after a tedious computation, one finds that the energy-momentum tensor satisfies

$$\begin{aligned}\nabla^\mu T_{\mu\nu} \;=\;& (\mathrm{d}A)_{\nu\mu}J^\mu_{\rm R} - \frac{3}{2}(\mathrm{d}V)_{\nu\mu}J^\mu_{\rm FZ} + (\mathrm{d}K)_{\nu\mu}J^\mu_K \\ &+\frac{3}{2}V_\nu\nabla_\mu J^\mu_{\rm FZ} - K_\nu\nabla_\mu J^\mu_K \;,\end{aligned} \tag{3.73}$$

where $J^\mu_{\rm FZ}$ is the Ferrara-Zumino current

$$J^\mu_{\rm FZ} \;=\; -\frac{2}{3}\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta V_\mu} \;, \tag{3.74}$$

and

$$J^\mu_K = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta K_\mu} \;. \tag{3.75}$$

Neither $J^\mu_{\rm FZ}$ nor J^μ_K are conserved. Explicit expressions for $T_{\mu\nu}$, $J^\mu_{\rm FZ}$, and J^μ_K are given in appendix C. Note that in this context, we must formally treat K_μ as a background field, although it was introduced in the Lagrangian as a shift of the original fields A_μ and V_μ. For the usual chiral multiplet Lagrangian with $\epsilon = 1$, however, one has $J^\mu_K = 0$.

For a generic Killing vector ξ, that is also a symmetry of the background fields, $\mathcal{L}_\xi A = \mathcal{L}_\xi V = 0$, we can define a conserved current as

$$Y^\mu_\xi \;=\; \xi_\nu\left(T^{\mu\nu} + J^\mu_{\rm R}A^\nu - \frac{3}{2}J^\mu_{\rm FZ}V^\nu + J^\mu_K K^\nu\right) \;. \tag{3.76}$$

One can show that indeed $\nabla_\mu Y^\mu_\xi = 0$. In particular, for $\xi = \partial_t$, one finds that the conserved charge is the Hamiltonian density

$$\mathcal{H} \;=\; -Y^t_{\partial_t} \;, \tag{3.77}$$

up to a total derivative on the three-sphere.

3.4.2 Canonical quantization

We now expand the dynamical fields in terms of creation and annihilation operators. Let us first focus on the scalar field. In order for the field ϕ to solve its equation of motion, we expand it as

$$\phi(x) \;=\; \sum_{\ell=0}^{\infty}\sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\left(a_{\ell mn}u^{(+)}_{\ell mn}(x) + b^\dagger_{\ell mn}(u^{(-)}_{\ell mn})^*(x)\right) \;, \tag{3.78}$$

with[10]

$$u^{(\pm)}_{\ell mn}(x) \;\equiv\; \frac{1}{4\sqrt{\omega^\pm_{\ell m} \mp \mu}}\mathrm{e}^{-\mathrm{i}\omega^\pm_{\ell m}t}Y^{mn}_\ell(\vec{x}) \;, \tag{3.79}$$

where $Y^{mn}_\ell(\vec{x})$ are the scalar spherical harmonics on a three-sphere of unit radius (see appendix A for further details), and

$$\omega^\pm_{\ell m} \;=\; \pm\mu + \sqrt{\frac{\alpha_b}{2}\ell(\ell+2) \pm 2\beta_b m + \gamma_b + \mu^2} \;. \tag{3.80}$$

[10] Although none the eigenvalues relevant to us depend on the $SU(2)_r \subset SO(4)$ quantum number n, we keep track of this in the spherical harmonics and in the expansions.

The canonical commutation relations

$$\begin{aligned} [\phi(t,\vec{x}),\Pi(t,\vec{x}\,')] &= \frac{\mathrm{i}}{\sqrt{-g}}\delta^{(3)}(\vec{x}-\vec{x}\,') \ , \\ [\phi(t,\vec{x}),\phi(t,\vec{x}\,')] &= [\Pi(t,\vec{x}),\Pi(t,\vec{x}\,')] \ = \ 0 \ , \end{aligned} \tag{3.81}$$

with $\delta^{(3)}(\vec{x}-\vec{x}\,') = \delta(\theta-\theta')\delta(\varphi-\varphi')\delta(\varsigma-\varsigma')$, hold by taking the oscillators to satisfy the usual

$$[a_{\ell mn}, a^\dagger_{\ell' m' n'}] \ = \ [b_{\ell mn}, b^\dagger_{\ell' m' n'}] = \delta_{\ell,\ell'}\delta_{m,m'}\delta_{n,n'} \ . \tag{3.82}$$

From (3.60) it follows that the Hamiltonian of the scalar field reads

$$\begin{aligned} H_{\mathrm{bos}} \ &= \ \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \omega^+_{\ell m}\left(a_{\ell mn}a^\dagger_{\ell mn} + a^\dagger_{\ell mn}a_{\ell mn}\right) \\ &\quad + \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \omega^-_{\ell m}\left(b_{\ell mn}b^\dagger_{\ell mn} + b^\dagger_{\ell mn}b_{\ell mn}\right) \ . \end{aligned} \tag{3.83}$$

Notice that we have used the *Weyl ordering* prescription, as this is the correct one for comparison with the path integral approach.

For the fermion, we expand the field ψ in terms of the spinor spherical harmonics $\mathbf{S}^\pm_{\ell mn}$. As discussed in appendix A, these are eigenspinors of the operator $\mathcal{O}_f$,

$$\mathcal{O}_f\mathbf{S}^\pm_{\ell mn} \ = \ \lambda^\pm_{\ell m}\mathbf{S}^\pm_{\ell mn} \ , \tag{3.84}$$

with the eigenvalues $\lambda^\pm_{\ell m}$ given in equation (3.33). In addition, there are the "special" spherical harmonics,

$$\mathcal{O}_f\mathbf{S}^{\mathrm{special}\pm}_{\ell n} \ = \ \lambda^{\mathrm{special}\pm}_{\ell}\mathbf{S}^{\mathrm{special}\pm}_{\ell n} \ , \tag{3.85}$$

with $\lambda^{\mathrm{special}\pm}_{\ell}$ given in equation (3.34). We expand the field ψ as

$$\psi_\alpha \ = \ \sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}} c_{\ell mn}u_{\ell mn\,\alpha} + \sum_{\ell=1}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1} d^\dagger_{\ell mn}v_{\ell mn\,\alpha} \ , \tag{3.86}$$

with

$$u_{\ell mn\,\alpha}(x) \ = \ \frac{1}{2\sqrt{2}}\mathrm{e}^{\mathrm{i}t\lambda^-_{\ell m}}\mathbf{S}^-_{\ell mn\,\alpha}(\vec{x}) \ , \qquad v_{\ell mn\,\alpha}(x) \ = \ \frac{1}{2\sqrt{2}}\mathrm{e}^{\mathrm{i}t\lambda^+_{\ell m}}\mathbf{S}^+_{\ell mn\,\alpha}(\vec{x}) \ . \tag{3.87}$$

Here we included $\mathbf{S}^{\mathrm{special}\pm}$ in the sums by defining

$$\begin{aligned} \mathbf{S}^-_{\ell,\frac{\ell}{2},n} \ &\equiv \ \mathbf{S}^{\mathrm{special}+}_{\ell n} \ , & \lambda^-_{\ell,\frac{\ell}{2},n} \ &\equiv \ \lambda^{\mathrm{special}+}_{\ell n} \ , \\ \mathbf{S}^-_{\ell,-\frac{\ell}{2}-1,n} \ &\equiv \ \mathbf{S}^{\mathrm{special}-}_{\ell n} \ , & \lambda^-_{\ell,-\frac{\ell}{2}-1,n} \ &\equiv \ \lambda^{\mathrm{special}-}_{\ell n} \ . \end{aligned} \tag{3.88}$$

Of course, by imposing the anti-commutation relations

$$\{c_{\ell mn}, c^{\dagger}_{\ell mn}\} \;=\; \{d_{\ell mn}, d^{\dagger}_{\ell mn}\} \;=\; \delta_{\ell,\ell'}\delta_{m,m'}\delta_{n,n'} \;, \tag{3.89}$$

one finds the field ψ_α and the conjugate momentum $\pi^\alpha = \mathrm{i}\widetilde{\psi}_{\dot\alpha}\widetilde{\sigma}^{0\,\dot\alpha\alpha}$ satisfy the canonical relations

$$\begin{aligned}
\{\psi_\alpha(t,\vec{x}), \pi^\beta(t,\vec{x}\,')\} &= \frac{\mathrm{i}}{\sqrt{-g}}\delta^{(3)}(\vec{x}-\vec{x}\,')\,\delta_\alpha{}^\beta \;, \\
\{\psi_\alpha(t,\vec{x}), \psi_\beta(t,\vec{x}\,')\} &= \{\pi^\alpha(t,\vec{x}), \pi^\beta(t,\vec{x}\,')\} \;=\; 0 \;.
\end{aligned} \tag{3.90}$$

The mode expansion (3.86) can now be inserted into the conserved charges of the previous subsection, recalling that these have to be Weyl ordered. For example, the Hamiltonian density in (3.60) becomes

$$\mathcal{H}_{\text{fer}} \;=\; \frac{1}{2}\big((\mathcal{O}_f\psi)\widetilde{\psi} - \widetilde{\psi}\mathcal{O}_f\psi\big) \;. \tag{3.91}$$

Inserting the mode expansion and integrating over the S^3 yields the quantized Hamiltonian

$$\begin{aligned}
H_{\text{fer}} \;=\;& \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}} \lambda^-_{\ell m}\left(c_{\ell mn}c^{\dagger}_{\ell mn} - c^{\dagger}_{\ell mn}c_{\ell mn}\right) \\
& -\frac{1}{2}\sum_{\ell=1}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1} \lambda^+_{\ell m}\left(d_{\ell mn}d^{\dagger}_{\ell mn} - d^{\dagger}_{\ell mn}d_{\ell mn}\right) \;.
\end{aligned} \tag{3.92}$$

In the next subsection we will turn to the computation of the expectation values of these Hamiltonians, and we will show that the infinite sums can be evaluated with (Hurwitz) zeta function regularization in two special cases. One case is obtained for $\mathfrak{q} = 0$, for which we can use the pairing of bosonic and fermionic eigenvalues discussed in section 3.3 to evaluate the vev of $H = H_{\text{bos}} + H_{\text{fer}}$. Another case is obtained for $\beta_f = \beta_b = 0$, where we will be able to evaluate the vevs of H_{bos} and H_{fer} separately.

Thus, for simplicity in the remainder of this subsection we restrict to $\beta_f = \beta_b = 0$. Using the mode expansions of the fields, and after Weyl ordering, we obtain expressions for the remaining conserved charges. For the R-charge, equation (3.64), this leads to

$$
\begin{aligned}
R &= \frac{r}{2}\sum_{\ell=0}\sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\left(a_{\ell mn}a^\dagger_{\ell mn}+a^\dagger_{\ell mn}a_{\ell mn}\right)-\frac{r}{2}\sum_{\ell=0}\sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\left(b^\dagger_{\ell mn}b_{\ell mn}+b_{\ell mn}b^\dagger_{\ell mn}\right)\\
&\quad -\frac{r-1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}}\left(c_{\ell mn}c^\dagger_{\ell mn}-c^\dagger_{\ell mn}c_{\ell mn}\right)\\
&\quad +\frac{r-1}{2}\sum_{\ell=1}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1}\left(d_{\ell mn}d^\dagger_{\ell mn}-d^\dagger_{\ell mn}d_{\ell mn}\right)\ . \qquad (3.93)
\end{aligned}
$$

For the J_3 angular momentum, equation (3.65), we get

$$
\begin{aligned}
J_3 &= \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} m\left(a_{\ell mn}a^\dagger_{\ell mn}+a^\dagger_{\ell mn}a_{\ell mn}\right)\\
&\quad +\frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} m\left(b^\dagger_{\ell mn}b_{\ell mn}+b_{\ell mn}b^\dagger_{\ell mn}\right)\\
&\quad -\frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}}\left(m+\frac{1}{2}\right)\left(c_{\ell mn}c^\dagger_{\ell mn}-c^\dagger_{\ell mn}c_{\ell mn}\right)\\
&\quad +\frac{1}{2}\sum_{\ell=1}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1}\left(m+\frac{1}{2}\right)\left(d_{\ell mn}d^\dagger_{\ell mn}-d^\dagger_{\ell mn}d_{\ell mn}\right)\ , \qquad (3.94)
\end{aligned}
$$

and finally the supercharge (3.68) reads

$$
\begin{aligned}
\mathcal{Q} &= -\mathrm{i}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sqrt{\frac{\ell}{2}+m+1}\ a^\dagger_{\ell mn}c_{\ell mn}\\
&\quad -\mathrm{i}\sum_{\ell=1}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}\sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1}(-1)^{-m-n}\sqrt{\frac{\ell}{2}-m}\ b_{\ell,-m,-n}d^\dagger_{\ell mn}\ . \qquad (3.95)
\end{aligned}
$$

By direct computation, one can now verify the following commutation relations

$$
[H,\mathcal{Q}] = -\frac{\mathfrak{q}}{r_3}\mathcal{Q}\ , \qquad [R,\mathcal{Q}] = \mathcal{Q}\ , \qquad [J_3,\mathcal{Q}] = -\frac{1}{2}\mathcal{Q}\ , \qquad (3.96)
$$

where we restored the radius r_3 of the S^3. Note that the Hamiltonian commutes with $\mathcal{Q}$ only for $\mathfrak{q}=0$, which from equation (3.37) is the value required for the pairing of eigenvalues. By conjugating equation (3.95), one can further verify that

$$
\frac{r_3}{2}\{\mathcal{Q},\mathcal{Q}^\dagger\} = :H+\frac{1}{r_3}(1+\mathfrak{q})R+\frac{2}{r_3}J_3:\ , \qquad (3.97)
$$

where : : denotes normal ordering. Let us set $r_3 = 1$ in (3.97) and (3.96) and comment on the special values of the parameter $\mathfrak{q}$ discussed in the literature. Setting $\mathfrak{q} = 0$ we have

$$\begin{aligned} \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\} &= \; : H + R + 2J_3 : \;, \\ [H, \mathcal{Q}] &= 0 \;, \end{aligned} \tag{3.98}$$

corresponding[11] to the relations in equation (5.9) of [20], where $H|_{\mathfrak{q}=0}$ coincides with H in that reference. For this reason we refer to $H|_{\mathfrak{q}=0} \equiv H_{\text{susy}}$ as the BPS Hamiltonian.

Setting $\mathfrak{q} = 1/2$ we have

$$\begin{aligned} \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\} &= \; : H + \frac{3}{2}R + 2J_3 : \;, \\ [H, \mathcal{Q}] &= -\frac{1}{2}\mathcal{Q} \;, \end{aligned} \tag{3.99}$$

which coincide for example with equation (7) of [57] as well as with equation (6.11) in [20], where $H|_{\mathfrak{q}=1/2}$ corresponds to Δ in the latter reference.

Finally, setting $\mathfrak{q} = -1$ we have

$$\begin{aligned} \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\} &= \; : H + 2J_3 : \;, \\ [H, \mathcal{Q}] &= \mathcal{Q} \;, \end{aligned} \tag{3.100}$$

corresponding to equation (5.6) of [20], where $H|_{\mathfrak{q}=-1}$ corresponds to P_0 in that reference.

Although these commutation relations are here written for the chiral multiplet, it is straightforward to verify that they hold also for the vector multiplet, and hence for the total $H_{\text{tot}} = H + H_{\text{vec}}$, and similarly for the other operators. It was noticed in [68] that these may be formally derived from the abstract supersymmetry algebra of new minimal supergravity.

3.4.3 Casimir energy

We are now ready to compute the vacuum expectation value of the Hamiltonian. This yields infinite sums which we regularize using the zeta function method. Thus, for an operator A, we define its vacuum expectation value as

$$\langle A \rangle \equiv \lim_{s \to -1} \zeta_A(s) \;, \tag{3.101}$$

[11] Here and below, the equations correspond to those referenced, up to convention dependent signs of R and J_3, as well a possible factor $\sqrt{2}$ in the supercharge $\mathcal{Q}$, descending from the definition of the supersymmetry variations.

where, denoting with λ_n^A the set of all the eigenvalues (here n is a multi-index) of A and with d_n^A their degeneracies, the generalized zeta function is defined as

$$\zeta_A(s) \;=\; \mathrm{Tr}\, A^{-s} \;=\; \sum_n d_n^A (\lambda_n^A)^{-s} \,. \tag{3.102}$$

Notice that if $A = B + C$, with corresponding eigenvalues denoted as λ_n^B and λ_n^C, then

$$\lim_{s\to -1} \left(\sum_n (\lambda_n^B)^{-s} + \sum_n (\lambda_n^C)^{-s} \right) \;\neq\; \lim_{s\to -1} \sum_n (\lambda_n^B + \lambda_n^C)^{-s} \,. \tag{3.103}$$

This lack of additivity is related to the lack of associativity of functional determinants, $\det(BC) \neq \det(B)\cdot\det(C)$, which is known as "multiplicative anomaly". See *e.g.* [100].

In the present context, we use the following prescription for dealing with the infinite sums: for each given operator, we sum independently the eigenvalues corresponding to every different field. In particular, we define the vev of each operator as the sum of the vevs of the terms containing the fields ϕ, ψ, A, and λ, respectively. Therefore, for example,

$$\langle H \rangle \;\equiv\; \langle H_{\mathrm{bos}} \rangle + \langle H_{\mathrm{fer}} \rangle \,, \tag{3.104}$$

and similarly for R, J_3, and $\mathcal{Q}$. This recipe is in accordance with [101], and we will show below that this yields the supersymmetric Casimir energy computed in [26].

The vevs of the scalar and fermion Hamiltonians of the chiral multiplet, (3.83) and (3.92), are

$$\begin{aligned} \langle H_{\mathrm{bos}} \rangle \;&=\; \lim_{s\to -1} \left[\frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}} (\omega_{\ell m}^{+})^{-s} + \frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{m,n=-\frac{\ell}{2}}^{\frac{\ell}{2}} (\omega_{\ell m}^{-})^{-s} \right] \,, \\ \langle H_{\mathrm{fer}} \rangle \;&=\; \lim_{s\to -1} \left[\frac{1}{2} \sum_{\ell=0}^{\infty} \sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}} (\lambda_{\ell m}^{-})^{-s} - \frac{1}{2} \sum_{\ell=1}^{\infty} \sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1} (\lambda_{\ell m}^{+})^{-s} \right] \,, \end{aligned} \tag{3.105}$$

respectively. However, due to the square roots appearing in both sets of eigenvalues $\omega_{\ell m}^{\pm}$ and $\lambda_{\ell m}^{\pm}$, the vevs in (3.105) cannot in general be separately regularized with any[12] zeta function and written in closed form.

In the special case $\mathfrak{q} = 0$, we can take advantage of the pairing as discussed in section 3.3 to compute the vev of the Hamiltonian of the chiral multiplet, $H =$

[12] *E.g.* Hurwitz, Barnes, Shintani, Epstein zeta functions.

$H_{\rm bos}+H_{\rm fer}$. Thus, setting $\mathfrak{q}=0$ one has

$$\omega^+_{\ell m} \;=\; -\lambda^-_{\ell m}\ , \qquad \omega^-_{\ell,-m} \;=\; \lambda^+_{\ell m}\ , \qquad \text{for} \quad \ell\geq 1\ , \quad -\frac{\ell}{2}\leq m\leq \frac{\ell}{2}-1\ . \tag{3.106}$$

The eigenvalues not included in equation (3.106) are the "special" fermion eigenvalues, which we can write as

$$\lambda^{\rm special\pm}_{\ell} \;=\; -\frac{1}{2}(\ell+1)\pm\beta_f-\mu\ , \qquad \ell\geq 0\ , \tag{3.107}$$

and the "unpaired" bosonic eigenvalues

$$\omega^+_{\ell,\frac{\ell}{2}} \;=\; \frac{1}{2}(\ell+1)-\beta_f+\mu\ , \qquad \omega^-_{\ell,-\frac{\ell}{2}} \;=\; \frac{1}{2}(\ell+1)-\beta_f-\mu\ , \qquad \ell\geq 0\ . \tag{3.108}$$

Here we used that $\alpha_f<0$ and assumed $\beta_f\leq -\frac{\alpha_f}{2}$ in order simplify the square roots in $\omega^+_{\ell,\frac{\ell}{2}}$ and $\omega^-_{\ell,-\frac{\ell}{2}}$. Due to the pairing, equation (3.106), all eigenvalues containing square roots exactly cancel against each other in (3.104), and we are left with

$$\begin{aligned}
\langle H\rangle_{\mathfrak{q}=0} \;&=\; \lim_{s\to-1}\left[\frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}(\omega^+_{\ell,\frac{\ell}{2}})^{-s}+\frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}(\omega^-_{\ell,-\frac{\ell}{2}})^{-s}\right.\\
&\qquad\left.+\frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}(\lambda^{\rm special+}_{\ell})^{-s}+\frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}}(\lambda^{\rm special-}_{\ell})^{-s}\right]\\
&=\; \lim_{s\to-1}\left[\frac{1}{4}\sum_{k=1}^{\infty}k\,(k-2(\beta_f+\mu))^{-s}-\frac{1}{4}\sum_{k=1}^{\infty}k\,(k+2(\beta_f+\mu))^{-s}\right]\\
&=\; \frac{1}{12}(\beta_f+\mu)\left(1-8(\beta_f+\mu)^2\right)\ .
\end{aligned}\tag{3.109}$$

Notice that the first and third term in the first line further exactly cancelled and in the last step we regularized *separately* the two remaining sums using the Hurwitz zeta function[13]. To summarize, since for $\mathfrak{q}=0$ one has $2(\beta_f+\mu)=1-r$, the vev of the Hamiltonian of a chiral multiplet with R-charge r is

$$\langle H\rangle_{\mathfrak{q}=0} \;=\; \frac{1}{12r_3}(1-r)\left(1-2(1-r)^2\right)\ , \tag{3.110}$$

where we restored r_3. This result is valid for any value of r, κ, and ϵ. Notice that if we were to combine the two sums in the middle line of (3.109), before regularization, we would get a different result.

Turning to the vector multiplet, the Casimir energy of the gauge field A does not depend on any of our parameters and is simply given by the result for an Abelian gauge field $\langle H_{\rm gauge}\rangle=\frac{11}{120r_3}$ (see *e.g.* [42, 102]) multiplied by the dimension

[13]See appendix B for details on the Hurwitz zeta function.

of the gauge group N_v. For the gaugino λ, the Casimir energy is computed as for the fermion ψ in equation (3.105), but using the eigenvalues of the operator $\mathcal{O}_f^{\rm vec}$ in equation (3.26). For $\mathfrak{q} = 0$ this gives simply $\langle H_{\rm gaugino}\rangle = -\frac{1}{120 r_3}$, again to be multiplied by N_v. It is now simple to combine this with the contributions from the chiral multiplets and vector multiplet, and recover[14] the supersymmetric Casimir energy $E_{\rm susy}$ in equation (3.51),

$$\langle H_{\rm tot}\rangle_{\mathfrak{q}=0} \;=\; \frac{1}{12 r_3}\big(2{\rm tr}\mathbf{R}^3 - {\rm tr}\mathbf{R}\big) \;=\; \frac{4}{27 r_3}(\mathbf{a}+3\mathbf{c}) \;=\; \frac{1}{r_3}E_{\rm susy}\ . \tag{3.111}$$

As in (3.51), this result is valid for arbitrary values of κ and ϵ. Indeed this is exactly the same BPS Hamiltonian defining the path integral, and therefore the free field result should have agreed with the localization result, that is valid for any value of the couplings.

Next, we consider the special case $\beta_b = \beta_f = 0$. This corresponds to setting $\kappa = -1$ and $\epsilon = 1$, but leaving arbitrary $\mathfrak{q}$. In this case, both sums in (3.105) can be separately regularized using Hurwitz zeta function, as the square roots in $\omega^{\pm}_{\ell m}$ and $\lambda^{\pm}_{\ell m}$ are absent, namely

$$\omega^+_\ell \;=\; \frac{1}{2}(\ell + 2 - r(1+\mathfrak{q}))\ , \qquad \omega^-_\ell \;=\; \frac{1}{2}(\ell + r(1+\mathfrak{q}))\ , \tag{3.112}$$

and

$$\lambda^-_\ell \;=\; \lambda^{\rm special\pm}_\ell \;=\; -\frac{1}{2}(\ell+2-r(1+\mathfrak{q})+\mathfrak{q})\ , \qquad \lambda^+_\ell \;=\; \frac{1}{2}(\ell+r(1+\mathfrak{q})-\mathfrak{q})\ , \tag{3.113}$$

where we dropped the subscript m, as this quantum number becomes degenerate. Thus, regularizing the sums as described at the beginning of this subsection using the Hurwitz zeta function, we obtain the finite Casimir energies

$$\langle H_{\rm bos}\rangle \;=\; \frac{1}{240}\Big[1 - 10\big(r(1+\mathfrak{q})-1\big)^4\Big]\ , \tag{3.114}$$

and

$$\begin{aligned}\langle H_{\rm fer}\rangle \;=\; \frac{1}{240}\Big[&10(\mathfrak{q}+1)^3\,(1+\mathfrak{q})\,(r-1)^4 \\ &+20(\mathfrak{q}+1)^3(r-1)^3 - 10(\mathfrak{q}+1)(r-1) - 1\Big]\ .\end{aligned} \tag{3.115}$$

Adding (3.114) and (3.115), we obtain the Casimir energy of a chiral multiplet

[14] The quantity $E_{\rm susy}$ defined in [26] is dimensionless. Therefore, when writing the radius of the three-sphere explicitly, this has to be compared with the dimensionless combination $r_3\langle H_{\rm tot}\rangle_{\mathfrak{q}=0}$.

with R-charge r

$$\begin{aligned}\langle H\rangle &= -\frac{1}{24}\Big[\mathfrak{q}^4+2(\mathfrak{q}+1)^3(2\mathfrak{q}-1)(r-1)^3\\ &\quad +6\mathfrak{q}^2(\mathfrak{q}+1)^2(r-1)^2+(\mathfrak{q}+1)\left(4\mathfrak{q}^3+1\right)(r-1)\Big]\,. \end{aligned}\tag{3.116}$$

This generalizes straightforwardly to an arbitrary number of chiral multiplets. As before, we can include easily an arbitrary number N_v of vector multiplets as well. In this case, for the gaugino, the Casimir energy can be obtained by formally setting $r=\frac{2\mathfrak{q}}{1+\mathfrak{q}}$ in equation (3.115), and reads

$$\langle H_{\text{gaugino}}\rangle = \frac{1}{240}\left(10\left(\mathfrak{q}^4-2\mathfrak{q}^3+\mathfrak{q}\right)-1\right).\tag{3.117}$$

Combining these results, we find that (for $\kappa=-1$, $\epsilon=1$) using our regularization, the Casimir energy of a supersymmetric gauge theory with N_v vector multiplets and N_χ chiral multiplets with R-charges r_I is given by the following expression

$$\begin{aligned}\langle H_{\text{tot}}\rangle &= \frac{N_v}{12r_3}\left(\mathfrak{q}^4-2\mathfrak{q}^3+\mathfrak{q}+1\right)-\frac{1}{12r_3}\sum_{I=1}^{N_\chi}\Big(\mathfrak{q}^4+\left(4\mathfrak{q}^3+1\right)(\mathfrak{q}+1)(r_I-1)\\ &\quad +6\mathfrak{q}^2(\mathfrak{q}+1)^2(r_I-1)^2+2(2\mathfrak{q}-1)(\mathfrak{q}+1)^3(r_I-1)^3\Big)\,,\end{aligned}\tag{3.118}$$

where we restored the radius r_3 of the three-sphere. Setting $\mathfrak{q}=0$ as in [26], and recalling the definition (3.52) of the anomaly coefficients $\mathbf{a}$ and $\mathbf{c}$, we see that (3.118) reduce to

$$\langle H_{\text{tot}}\rangle_{\mathfrak{q}=0} = \frac{4}{27r_3}(\mathbf{a}+3\mathbf{c})\,,\tag{3.119}$$

in agreement with (3.111).

In general, however, equation (3.118) cannot be written as a linear combination of $\mathbf{a}$ and $\mathbf{c}$. In the special case $\mathfrak{q}=1/2$ and $r_I=2/3$, corresponding to the usual conformally coupled scalars, Weyl spinors, and gauge fields, the Casimir energy (3.118) reduces to

$$\langle H_{\text{tot}}\rangle_{\mathfrak{q}=\frac{1}{2},r_I=\frac{2}{3}} = \frac{1}{192r_3}(21N_v+5N_\chi) = \frac{1}{4r_3}(\mathbf{a}+2\mathbf{c})\,,\tag{3.120}$$

in accordance with standard zeta function computations (see *e.g.* [42][15]). In particular, notice that for theories with $N_\chi=3N_v$ so that $\mathbf{a}=\mathbf{c}$, such as $\mathcal{N}=4$ super-Yang Mills, this becomes simply $\frac{3}{4r_3}\mathbf{a}$. However, the agreement with the CFT result of [103] for the Casimir energy is accidental [102, 103]. Finally, we note that

[15] Equation (5.60) of [42] gives the Casimir energy of n_0 scalar, $n_{1/2}$ Weyl fermions, and n_1 vector fields. Agreement with (3.120) is found setting $n_0=2N_\chi$, $n_{1/2}=N_v+N_\chi$, and $n_1=N_v$.

for $\mathfrak{q} = -1$, the Casimir energy is independent of the R-charges and reads

$$\langle H_{\rm tot} \rangle_{\mathfrak{q}=-1} \;\; = \;\; \frac{1}{12 r_3} \left(3 N_v - N_\chi \right) \; . \qquad (3.121)$$

This is simply because in this case $A = 0$ from equation (3.10), and therefore the Lagrangian does not depend on the R-charges.

We can also compute the vevs of the supercharge $\mathcal{Q}$ and R-symmetry charge R. It is simple to see from its mode expansion (3.95) that $\mathcal{Q}$ annihilates the vacuum, and so the vev of $\mathcal{Q}$ is zero. The same is true for the supercharge of the vector multiplet, which is explicitly given by $\mathcal{Q}_{\rm vec} = \frac{\rm i}{2} \int \sqrt{g_3} {\rm d}^3 x \; \zeta \sigma^\mu \widetilde{\sigma}^\nu \sigma^0 \widetilde{\lambda} \mathsf{F}_{\mu\nu}$. From the mode expansion (3.93) of R for the chiral multiplet, it is clear that the scalar field does not contribute. Furthermore, since the fermionic eigenvalues are constant, they do not fit in the regularization scheme of eqns. (3.101) and (3.102) as they do not give rise to a zeta function. In [2], the regularization proceeds by noting that only the eigenvalues from the "special" modes, for which $m = -\frac{\ell}{2} - 1$ and $m = \frac{\ell}{2}$, do not cancel,

$$\begin{aligned} \langle R \rangle \;\; &= \;\; -\frac{r-1}{2} \sum_{\ell=0}^{\infty} \sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \sum_{m=-\frac{\ell}{2}-1}^{\frac{\ell}{2}} 1 + \frac{r-1}{2} \sum_{\ell=1}^{\infty} \sum_{n=-\frac{\ell}{2}}^{\frac{\ell}{2}} \sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}-1} 1 \\ &= \;\; (1-r) \sum_{\ell=0}^{\infty} (\ell+1) \\ &= \;\; \frac{r-1}{12} \; , \end{aligned} \qquad (3.122)$$

where in the last step we used the Riemann zeta function, $\zeta_R(-1) = -\frac{1}{12}$. Similarly, for the vector multiplet $\langle R_{\rm vec} \rangle = \frac{1}{12}$, where only the gaugino contributes. Thus, the vev of the total R-charge operator $R_{\rm tot} = R + R_{\rm vec}$ presented in [2] is

$$\langle R_{\rm tot} \rangle \;\; = \;\; \frac{4}{3} (\mathbf{a} - \mathbf{c}) \; . \qquad (3.123)$$

It was clarified in [3], however, that this regularization of $\langle R \rangle$ does not respect supersymmetry. As we will see below in section 3.5, the correct regularization yields $\langle R \rangle = -r_3 \langle H_{\rm susy} \rangle$, where $H_{\rm susy} = H|_{\mathfrak{q}=0}$.

The results discussed in this section rely on the fact that the operators we are using are not normal ordered. See also [104] for a similar discussion.

3.5 Reduction to supersymmetric quantum mechanics

In this section we continue the study of the Casimir energy of $\mathcal{N}=1$ field theories on $\mathbb{R}\times S^3$, where the S^3 is round. We perform a manifestly supersymmetric analysis. By expanding all dynamical fields in spherical harmonics on the S^3, we reduce the problem to a supersymmetric quantum mechanical problem.

As discussed in section 1.5, the regularization of the partition function can be described in terms of the addition of counterterms to the action. In general, this can be done in different ways, leaving the finite part scheme-dependent and ambiguous. Different renormalization schemes differ by some local counterterm, that is, the addition to the action of some local term constructed from the background fields, $\int \mathrm{d}^4x\sqrt{g}\mathcal{L}_{\mathrm{ct}}(g_{\mu\nu}, A_\mu, V_\mu)$. In particular, dimensionless counterterms[16] affect the finite part of the computation, accounting for the ambiguity mentioned above. Below we argue that if the regularization is required to be compatible with supersymmetry, no such counterterm can shift the value of the supersymmetric Casimir energy.

This section is based on [3].

3.5.1 Consequences of the supersymmetry algebra

In the following we consider the special choice of the parameter $\kappa = \kappa^{\mathrm{st}} = -1$, discussed around (3.10). The background preserves four supercharges, and we set $\mathfrak{q}=0$ so that the supercharges are time-independent, see (3.11). Since the Hamiltonian commutes with the supercharges in this case (from equation (3.96)), we denote $H_{\mathrm{susy}} = H|_{\mathfrak{q}=0}$. Due to the flat gauge field A along the Euclidean time direction, the Hamiltonian H_{susy} is shifted with respect to what we would get from radial quantization as[17]

$$H_{\mathrm{susy}} \;=\; \Delta + \frac{1}{2r_3}R\;, \tag{3.124}$$

where Δ is the time translation operator obtained by mapping the dilatation operator in flat space to the cylinder.

The superalgebra preserved on this background [55] is

$$\begin{aligned}
\frac{r_3}{2}\{\mathcal{Q}_\alpha, \mathcal{Q}^{\dagger\beta}\} \;&=\; \delta^\beta{}_\alpha\Big(H_{\mathrm{susy}} + \frac{1}{r_3}R\Big) + \frac{2}{r_3}\gamma^{i\,\beta}{}_\alpha J^i_l \\
[H_{\mathrm{susy}}, \mathcal{Q}_\alpha] \;&=\; 0\;, \qquad [R, \mathcal{Q}_\alpha] \;=\; \mathcal{Q}_\alpha\;, \qquad [J^i_l, \mathcal{Q}_\alpha] \;=\; -\frac{1}{2}\mathcal{Q}_\beta\gamma^{i\,\beta}{}_\alpha\;,
\end{aligned} \tag{3.125}$$

where again γ^i are the Pauli matrices, R is the R-symmetry charge, J^i_l are the gen-

[16]That is, counterterms of the form $\int \mathrm{d}^4x\sqrt{g}\mathcal{L}_{\mathrm{ct}}(g_{\mu\nu}, A_\mu, V_\mu)$ where the integrand is of mass dimension four. Such counterterms are called *marginal* in [96].

[17]For consistency with the previous section, we changed here the sign of the R-charge operator R and angular momentum J^i compared to [3, 55].

erators of the $SU(2)_l \subset SU(2)_l \times SU(2)_r$ isometry of the sphere. The supercharges $\mathcal{Q}_\alpha$, $\alpha = 1, 2$ form a doublet under $SU(2)_l$, while the subgroup $SU(2)_r$ subgroup does not appear in the superalgebra.

A first remark is that we assume the vacuum $|\mathrm{VAC}\rangle$ does not break supersymmetry. Suppose the vacuum were not supersymmetric, in which case either $\mathcal{Q}_1$ or $\mathcal{Q}_2$ (or both) would not annihilate the vacuum. Then $\mathcal{Q}|\mathrm{VAC}\rangle$ is a new state with the same value of H_{susy}, but contributing with opposite sign to the index or partition function. Therefore if supersymmetry were broken, the index on $\mathbb{R} \times S^3$ would not receive a contribution from the unit operator. In the case of SCFTs, the fact that supersymmetry is unbroken on $\mathbb{R} \times S^3$ follows from radial quantization.

Another simple observation from (3.125) is that $J_3 = J_l^3$ annihilates the vacuum, $J_3|\mathrm{VAC}\rangle = 0$. Indeed, J_3 appears with different signs on the right hand side of $\{\mathcal{Q}_1, \mathcal{Q}_1^\dagger\}$ and $\{\mathcal{Q}_2, \mathcal{Q}_2^\dagger\}$. Hence, if both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ annihilates the vacuum, so must J_3.

It is useful to focus on the algebra of one specific supercharge, say $\mathcal{Q}_1$,

$$\begin{aligned} \frac{r_3}{2}\{\mathcal{Q}_1, \mathcal{Q}_1^\dagger\} &= H_{\text{susy}} + \frac{1}{r_3}R + \frac{2}{r_3}J_3 \ , \qquad \mathcal{Q}_1^2 \ = \ 0 \\ [H_{\text{susy}}, \mathcal{Q}_1] &= [R + 2J_3, \mathcal{Q}_1] \ = \ 0 \ . \end{aligned} \tag{3.126}$$

Since $\mathcal{Q}_1$ and J_3 annihilate the vacuum, the first line implies

$$\langle H_{\text{susy}} \rangle \ = \ -\frac{1}{r_3}\langle R \rangle \ . \tag{3.127}$$

Note the consistency of (3.126) with (3.96) and (3.97). This also shows that the regularization of $\langle R \rangle$ in the previous section leading to (3.123) does not respect supersymmetry, as we remarked at that point.

The supersymmetry algebra does not fix $\langle H_{\text{susy}} \rangle$ entirely, as (3.126) is invariant under shifts of H_{susy} and R. Our approach for determining $\langle H_{\text{susy}} \rangle$ will be to reduce the theory on the three-sphere. In this way, we obtain a quantum mechanics theory with infinitely many degrees of freedom. The theory has four supercharges $\mathcal{Q}_1$, $\mathcal{Q}_2$, and their Hermitian conjugates. The R-symmetry group is $SU(2)_l \times U(1)$ and the supercharges transform in the (2,1) representation. The $SU(2)_r$ symmetry is a global symmetry of the quantum mechanics theory.

3.5.2 Supersymmetric quantum mechanics

Let us model the situation in (3.126) as

$$\begin{aligned} \{\mathcal{Q}, \mathcal{Q}^\dagger\} &= 2(H + \Sigma) \ , \qquad \mathcal{Q}^2 \ = \ 0 \ , \\ [H, \mathcal{Q}] &= [\Sigma, \mathcal{Q}] \ = \ 0 \ , \end{aligned} \tag{3.128}$$

where H generates time translations, while Σ is some Hermitian conserved charge. We could just redefine H by Σ, however, in order to connect more easily with the reduction over S^3, we will keep the algebra in this form.

We can now define two types of multiplets: a *chiral multiplet* (ϕ, ψ), and a *Fermi multiplet* (λ, f), where ϕ,f are complex and commuting while ψ, λ are complex and anti-commuting. These two multiplets have the following supersymmetry transformations,

$$\begin{array}{lllll} \text{chiral} & : & \delta\phi \;=\; \sqrt{2}\zeta\psi \;, & & \delta\psi \;=\; -\sqrt{2}\mathrm{i}\zeta^\dagger D_t\phi \\ \text{Fermi} & : & \delta\lambda \;=\; \sqrt{2}\zeta f + p\sqrt{2}\zeta^\dagger\phi \;, & & \delta f \;=\; -\sqrt{2}\mathrm{i}\zeta^\dagger D_t\lambda - p\sqrt{2}\zeta^\dagger\psi \;, \end{array} \tag{3.129}$$

where on all the fields we define $D_t = \partial_t - \mathrm{i}\sigma$, with σ the charge of the field under Σ. The complex parameter ζ is independent of time and uncharged under Σ. In the variations of the Fermi multiplet there appears a parameter p. When $p = 0$, the chiral and Fermi multiplets are independent of each other. We will refer to each of the decoupled multiplets as "short". When instead $p \neq 0$ the two multiplets form one reducible but indecomposable representation of supersymmetry. Thus, for $p \neq 0$ we call the combined chiral and Fermi multiplets a "long" multiplet.

On each component of a multiplet with charge σ, the transformations (3.129) give

$$\{\delta_1, \delta_2\} \;=\; -2\mathrm{i}\big(\zeta_1^\dagger\zeta_2 + \zeta_2^\dagger\zeta_1\big)D_t \;, \tag{3.130}$$

which is consistent with the algebra (3.128) when H is represented as $-\mathrm{i}\partial_t$.

The supersymmetric Lagrangian of a long multiplet takes the form

$$\begin{aligned} L \;=\; & |D_t\phi|^2 - \mathrm{i}\nu(\phi D_t\psi^\dagger - \phi^\dagger D_t\phi) + \mathrm{i}\psi^\dagger D_t\psi - 2\nu\psi\psi^\dagger \\ & + \mathrm{i}\lambda^\dagger D_t\lambda + |f|^2 \\ & - p^2|\phi|^2 - p(\lambda\psi^\dagger + \psi\lambda^\dagger) \;, \end{aligned} \tag{3.131}$$

where ν is an additional free parameter, giving a mass to ψ. For $p = 0$, the first and second lines are the Lagrangians of a free chiral and Fermi multiplet, respectively, and are separately supersymmetric.

We now pass to Hamiltonian formalism and quantize the theory. The canonical momenta are

$$\Pi_\phi \;=\; (D_t + \mathrm{i}\nu)\phi^\dagger \;, \qquad \Pi_\psi \;=\; -\mathrm{i}\psi^\dagger \;, \qquad \Pi_\lambda \;=\; -\mathrm{i}\lambda^\dagger \;, \qquad \Pi_f \;=\; 0 \;. \tag{3.132}$$

The canonical (anti-)commutation relations are

$$[\phi, \Pi_\phi] \;=\; \mathrm{i} \;, \qquad \{\psi, \Pi_\psi\} \;=\; -\mathrm{i}\{\psi, \psi^\dagger\} \;=\; -\mathrm{i} \;, \qquad \{\lambda, \Pi_\lambda\} \;=\; -\mathrm{i}\{\lambda, \lambda^\dagger\} \;=\; -\mathrm{i} \;, \tag{3.133}$$

together with their Hermitian conjugates.

The Hamiltonian reads

$$\begin{aligned} H &= |\Pi_\phi|^2 + \mathrm{i}(\nu+\sigma)(\Pi_\phi\phi - \phi^\dagger\Pi_{\phi^\dagger}) + \nu^2|\phi|^2 + (\sigma+2\nu)\psi\psi^\dagger \\ &\quad + \sigma\lambda\lambda^\dagger \\ &\quad + p^2|\phi|^2 + p(\lambda\psi^\dagger + \psi\lambda^\dagger) + \widetilde{\alpha} \ , \end{aligned} \tag{3.134}$$

where again when $p = 0$ the first line gives the Hamiltonian of a chiral multiplet, while the second line is the Hamiltonian of a Fermi multiplet. The field f has been set to zero by its equation of motion. Note that we have introduced a constant $\widetilde{\alpha}$, parametrizing the usual ordering ambiguity.

In terms of canonical variables, the charge Σ reads

$$\Sigma \ = \ -\mathrm{i}\sigma\big(\Pi_\phi\phi - \phi^\dagger\Pi_{\phi^\dagger}\big) - \sigma\big(\psi\psi^\dagger + \lambda\lambda^\dagger\big) - \alpha \ , \tag{3.135}$$

where α parametrizes the ordering ambiguity in this operator. The supercharge is

$$\mathcal{Q} \ = \ \sqrt{2}\mathrm{i}\,\psi\big(\Pi_\phi - \mathrm{i}\nu\phi^\dagger\big) + \sqrt{2}\,p\,\phi^\dagger\lambda \ , \tag{3.136}$$

and is free of ordering ambiguities. Evaluating $\{\mathcal{Q}, \mathcal{Q}^\dagger\}$ we find that (3.128) is upheld provided we take

$$\widetilde{\alpha} \ = \ \alpha - 2\nu \ . \tag{3.137}$$

Hence supersymmetry fixes the ordering ambiguity in $H + \Sigma$. Of course, after having solved for $\widetilde{\alpha}$ we still have the freedom to shift H and Σ by an equal amount, corresponding to the remaining parameter α. Without additional assumptions, this freedom would have remained in the framework of ordinary quantum mechanics in one dimension.

In order to explain how to fix the ordering ambiguity that is left, it is useful to recall that we are computing the coefficient of a CS term in the low-energy $1d$ effective action. This term takes the form

$$k\int \mathrm{d}t\, A_t^\Sigma \ , \tag{3.138}$$

where A_t^Σ is the background gauge field associated to the charge Σ. A single fermion of mass m and charge q shifts the coefficient of the Chern-Simons term by $\frac{\mathsf{q}}{2}\mathrm{sgn}(\mathsf{m})$ [105]. We can think about this as if we are starting from some theory in the UV with Chern-Simons coefficient k_{uv} and then we integrate out the massive fermion leading to a Chern-Simons coefficient in the infrared k_{ir} (this interpretation was elaborated upon in [21])[18]

$$k_{\rm ir} - k_{\rm uv} \;=\; -\frac{\mathsf{q}}{2}\,{\rm sgn}(\mathsf{m}) \;. \tag{3.141}$$

From the point of view of the quantum mechanics, the arbitrariness in the charge of the vacuum corresponds to the arbitrariness in the UV coefficient $k_{\rm uv}$. However, our theory arises from a higher-dimensional model. As already observed, it is easy to convince oneself that a term like (3.138) cannot be generated by dimensional reduction of a four-dimensional local term. So we must take

$$k_{\rm uv} = 0 \;, \tag{3.142}$$

i.e. no Chern-Simons contact term in the UV generating functional. This key requirement fixes the ordering ambiguity in H. Together with (3.141), this implies that multiplets containing pairs of fermions with masses of opposite sign do not contribute to the Casimir energy. We will see below that as long as the Hamiltonian is bounded from below, a long multiplet necessarily contains fermions with masses of opposite sign. As a result, the choice of the ordering coefficient must be such that H and Σ vanish in the ground state of a long multiplet. This leads to the conclusion that the correct choice of the ordering constant is

$$\alpha = -2\sigma \;. \tag{3.143}$$

We will use this choice in the following and one can verify that in all cases the results are consistent with (3.142). Incidentally, it turns out that (3.143) also corresponds to Weyl ordering for the Hamiltonian.[19]

[18] A simple way to derive (3.141) is as follows. First, from dimensional analysis and the fact that m and k are odd under charge conjugation we infer

$$k_{\rm ir} - k_{\rm uv} \;=\; x\,{\rm sgn}(\mathsf{m}) \;, \tag{3.139}$$

where x is a coefficient, independent of m. To fix x we can consider a free fermion with mass m and charge q with a constant background gauge field A_t^Σ. This has Hamiltonian $H = (\mathsf{m}+\mathsf{q}A_t^\Sigma)(\psi\psi^\dagger + \hat\alpha)$, where $\hat\alpha$ is an arbitrary ordering constant. The partition function is given by

$$Z \;=\; e^{-\beta(\mathsf{m}+\mathsf{q}A_t^\Sigma)\hat\alpha}\left(1 + e^{-\beta(\mathsf{m}+\mathsf{q}A_t^\Sigma)}\right) \;. \tag{3.140}$$

The idea now is that we can keep the ultraviolet fixed and consider two different RG flows, one with positive m and one with negative m. By subtracting the resulting Chern-Simons terms in the infrared (which we will read out from the charge of the vacuum), we will find $2x$. If $\mathsf{m} > 0$ then taking $\mathsf{m} \to \infty$ we can read off the CS term (*i.e.* charge) in the IR to be $\mathsf{q}\hat\alpha \int {\rm d}t A_t^\Sigma$. On the other hand, if $\mathsf{m} < 0$ we read out the CS term in the IR by taking the limit $\mathsf{m} \to -\infty$ and we find $\mathsf{q}(\hat\alpha + 1)\int {\rm d}t A_t^\Sigma$. Subtracting these yields $2x = -\mathsf{q}$.

[19] This explains why the final result is identical to that of section 3.4 (from [2]) for the VEV of H. But, unlike [2], our result for the VEV of Σ in the vacuum manifestly respects the BPS condition $H = -\Sigma$.

3.5.3 Spectrum of the Hamiltonian

We now study the spectrum of the Hamiltonian (3.134) and determine the vacuum state.

Long multiplet

Let us start from the bosonic sector of (3.134):

$$H_{\rm bos} \;=\; |\Pi_\phi|^2 + \mathrm{i}(\nu+\sigma)(\Pi_\phi\phi - \phi^\dagger\Pi_{\phi^\dagger}) + (\nu^2+p^2)|\phi^2| - \nu - \sigma \;, \tag{3.144}$$

where we have included half of the ordering constant appearing there (the other half will enter in the fermionic sector). This ensures Weyl ordering. We can introduce creation operators $a^\dagger$, $b^\dagger$ and annihilation operators a, b via

$$\phi \;=\; \frac{(\nu^2+p^2)^{-1/4}}{\sqrt{2}}\big(a+b^\dagger\big) \;, \qquad \Pi_\phi \;=\; \frac{i(\nu^2+p^2)^{1/4}}{\sqrt{2}}\big(a^\dagger - b\big) \;. \tag{3.145}$$

The canonical commutation relations between ϕ and Π_ϕ (and their Hermitian conjugates) imply that these satisfy $[a,a^\dagger]=[b,b^\dagger]=1$, $[a,b]=[a^\dagger,b]=[a,b^\dagger]=[a^\dagger,b^\dagger]=0$. Then the bosonic Hamiltonian can be written as

$$\begin{aligned} H_{\rm bos} \;&=\; \sqrt{\nu^2+p^2}\big(a^\dagger a + b^\dagger b + 1\big) + (\sigma+\nu)\big(b^\dagger b - a^\dagger a\big) \\ &=\; \frac{1}{2}\sqrt{\nu^2+p^2}\big(\{a,a^\dagger\}+\{b,b^\dagger\}\big) + \frac{1}{2}(\sigma+\nu)\big(\{b,b^\dagger\}-\{a,a^\dagger\}\big) \;, \end{aligned} \tag{3.146}$$

where in the second line we have emphasized that $H_{\rm bos}$ is Weyl ordered. The state annihilated by a and b has energy $\sqrt{\nu^2+p^2}$. Acting on this with $(a^\dagger)^m(b^\dagger)^n$ (with m,n positive integers) we obtain a state with energy

$$H_{\rm bos}(m,n) \;=\; \sqrt{\nu^2+p^2} + m\big(\sqrt{\nu^2+p^2}-\nu-\sigma\big) + n\big(\sqrt{\nu^2+p^2}+\nu+\sigma\big) \;. \tag{3.147}$$

We see that in order for the Hamiltonian to have a spectrum that is bounded from below we need to assume $\sqrt{\nu^2+p^2} > |\nu+\sigma|$.[20] Hence the state of minimum energy in the bosonic sector is the one with $m=n=0$.

Next we address the fermionic sector. The Hamiltonian reads

$$\begin{aligned} H_{\rm fer} \;&=\; p(\lambda\psi^\dagger + \psi\lambda^\dagger) + (2\nu+\sigma)\psi\psi^\dagger + \sigma\lambda\lambda^\dagger - \nu - \sigma \\ &=\; \begin{pmatrix}\psi & \lambda\end{pmatrix}\begin{pmatrix} 2\nu+\sigma & p \\ p & \sigma\end{pmatrix}\begin{pmatrix}\psi^\dagger \\ \lambda^\dagger\end{pmatrix} - \nu - \sigma \;, \end{aligned} \tag{3.148}$$

where we have kept the ordering constant that ensures Weyl ordering. We can

[20] Allowing for $\sqrt{\nu^2+p^2} = |\nu+\sigma|$ yields a Hamiltonian bounded from below but introduces a degenerate vacuum. Let us discard this case.

make a unitary $U(2)$ rotation to diagonalize the above matrix. This preserves the anti-commutation relations. The eigenvalues are

$$x_{\pm} \;=\; \nu + \sigma \pm \sqrt{\nu^2 + p^2}\ . \tag{3.149}$$

Denoting the eigenvectors $u_+, u_-, u_+^\dagger, u_-^\dagger$, the Hamiltonian is thus

$$\begin{aligned} H_{\text{fer}} &= x_+ u_+ u_+^\dagger + x_- u_- u_-^\dagger - \nu - \sigma \\ &= \frac{x_+}{2}\,[u_+, u_+^\dagger] + \frac{x_-}{2}\,[u_-, u_-^\dagger]\ , \end{aligned} \tag{3.150}$$

with $\{u_\pm, u_\pm^\dagger\} = 1$. The charge operator Σ takes the form

$$\begin{aligned} \Sigma_{\text{fer}} &= \sigma\big(u_+ u_+^\dagger + u_- u_-^\dagger - 1\big) \\ &= \sigma\,[u_+, u_+^\dagger] + \sigma\,[u_-, u_-^\dagger]\ . \end{aligned} \tag{3.151}$$

Starting with the state $|0\rangle$ which is annihilated by both $u_\pm^\dagger$, we can act with u_-, u_+ or $u_- u_+$. The spectrum therefore consists of four states with the following energy and charge:

$$\begin{array}{r|cccc} \text{state} & |0\rangle & u_-|0\rangle & u_+|0\rangle & u_+u_-|0\rangle \\ \text{energy} & -\nu-\sigma & -\sqrt{\nu^2+p^2} & \sqrt{\nu^2+p^2} & \nu+\sigma \\ \text{charge} & -\sigma & 0 & 0 & \sigma \end{array} \tag{3.152}$$

Since we assumed $\sqrt{\nu^2 + p^2} > |\nu + \sigma|$, the state of lowest energy is $u_-|0\rangle$.

We now combine the information obtained studying the bosonic and fermionic sectors of the Hamiltonian and identify a state with minimum energy that respects supersymmetry. Adding H_{bos} and H_{fer}, the complete Hamiltonian is

$$\begin{aligned} H \;=\; & \sqrt{\nu^2 + p^2}\big(a^\dagger a + b^\dagger b + 1\big) + (\sigma + \nu)\big(b^\dagger b - a^\dagger a\big) \\ & + x_+ u_+ u_+^\dagger + x_- u_- u_-^\dagger - \nu - \sigma\ . \end{aligned} \tag{3.153}$$

One can also check that the full charge operator reads

$$\Sigma \;=\; -\sigma\left(b^\dagger b - a^\dagger a + u_+ u_+^\dagger + u_- u_-^\dagger - 1\right)\ . \tag{3.154}$$

From the discussion above, the state with minimum energy is clearly

$$|\text{VAC}\rangle \;\equiv\; |m = 0, n = 0, x_-\rangle\ , \tag{3.155}$$

where $m = 0$, $n = 0$ indicates that no bosonic oscillators are excited, and by x_- we mean that we excite one fermionic oscillator with eigenvalue x_-. Its total energy is

$$H \;=\; \sqrt{\nu^2+p^2} - \sqrt{\nu^2+p^2} \;=\; 0 \;, \tag{3.156}$$

and thus vanishes due to an exact cancellation between the bosonic and the fermionic contributions. Since we have just one fermionic oscillator the charge is $\Sigma = 0$, hence the relation $(H+\Sigma)|\mathrm{VAC}\rangle = 0$ is satisfied and supersymmetry is unbroken in the vacuum, as expected.

We conclude that the long multiplets yield a vanishing contribution to the vacuum energy and charge:

$$\langle H_{\text{long}}\rangle \;=\; \langle \Sigma_{\text{long}}\rangle \;=\; 0 \;. \tag{3.157}$$

Note that this is a consequence of our choice of ordering constant, and as argued at the end of the previous subsection this is the correct choice for a quantum mechanics arising from a higher-dimensional theory.

If we had a theory of long multiplets only, the vacuum energy would just be zero. However, if short multiplets are also present, this is not the case, as we now show.

Fermi multiplet

Consider the Fermi multiplet. Then the supercharge identically vanishes. The Hamiltonian and the charge generator take the same form,

$$H_{\text{Fermi}} \;=\; -\Sigma_{\text{Fermi}} \;=\; \sigma\Big(\lambda\lambda^\dagger - \frac{1}{2}\Big) \;. \tag{3.158}$$

The only two states have energy $-\frac{1}{2}\sigma$ and $+\frac{1}{2}\sigma$. The contribution of a Fermi multiplet to the vacuum energy and charge is thus

$$\langle H_{\text{Fermi}}\rangle \;=\; -\langle \Sigma_{\text{Fermi}}\rangle \;=\; -\frac{|\sigma|}{2} \;. \tag{3.159}$$

Chiral multiplet

The bosonic sector of the chiral multiplet can be treated as we did for the long multiplet, setting $p=0$. The full Hamiltonian and charge operator can thus be written as

$$H_{\text{chiral}} \;=\; |\nu|\big(a^\dagger a + b^\dagger b + 1\big) + (\sigma+\nu)\big(b^\dagger b - a^\dagger a\big) + (2\nu+\sigma)\psi\psi^\dagger - \nu - \frac{\sigma}{2} \;, \tag{3.160}$$

$$\Sigma_{\text{chiral}} \;=\; -\sigma\left(b^\dagger b - a^\dagger a\right) - \sigma\psi\psi^\dagger + \frac{1}{2}\sigma \;. \tag{3.161}$$

Since $p=0$, the condition for the Hamiltonian to be bounded from below becomes

$$|\nu| > |\nu+\sigma| \;. \tag{3.162}$$

In the vacuum all bosonic oscillators are zero. Then we have two possible states:

1. the state annihilated by $\psi^\dagger$, with $H = |\nu| - \nu - \frac{1}{2}\sigma$ and $\Sigma = +\frac{1}{2}\sigma$;
2. the state with an oscillator ψ excited, with $H = |\nu| + \nu + \frac{1}{2}\sigma$ and $\Sigma = -\frac{1}{2}\sigma$.

Which state has minimum energy depends on the values of ν and σ. Note that (3.162) requires ν and σ to have opposite signs. If $\nu > 0$, $\sigma < 0$, then (3.162) implies $-2\nu < \sigma < 0$, and the state number 1 has minimum energy $H = -\frac{1}{2}\sigma$; since $H = -\Sigma$, this state is supersymmetric, while the state 2 is non-supersymmetric. Conversely, if $\nu < 0$ and $\sigma > 0$, then from (3.162) we deduce $0 < \sigma < -2\nu$, hence the state number 1 now has higher energy and the state 2 is the supersymmetric vacuum, with $H = \frac{1}{2}\sigma$.

Thus, a chiral multiplet contributes to the vacuum energy and charge as

$$\langle H_{\text{chiral}} \rangle \;=\; -\langle \Sigma_{\text{chiral}} \rangle \;=\; \frac{|\sigma|}{2}\ . \tag{3.163}$$

In conclusion, the analysis in supersymmetric quantum mechanics establishes that a long multiplet yields a vanishing contribution to the vacuum energy and charge, that a Fermi multiplet contributes as in (3.159), while a chiral multiplet contributes as in (3.163).

3.5.4 Dimensional reduction of a 4d chiral multiplet

Consider a free four-dimensional chiral multiplet (ϕ, ψ, F) on $\mathbb{R} \times S^3$. The Lagrangian and supersymmetry transformations can be found in [20]. The only parameter appearing in the Lagrangian is the charge r under the background R-symmetry gauge field. Here we will restrict to $0 < r \leq 2$.[21] This range is compatible with the inequalities mentioned in the previous subsection, ensuring that the spectrum of the Hamiltonian is bounded from below. Expanding in appropriate spherical harmonics, the chiral multiplet reduces to a one-dimensional theory with infinitely many fields. These organize in one-dimensional multiplets with different values of the parameters ν, p, σ introduced above. Some have $p \neq 0$ and are thus long multiplets, while some others have $p = 0$ and are thus short multiplets, either chiral or Fermi.

More explicitly, we can expand the scalars in spherical harmonics Y_ℓ^{mn}, discussed in appendix A. The quantum number ℓ is a non-negative integer. For a fixed ℓ, the quantum numbers m, n of the scalar harmonic Y_ℓ^{mn} range in $-\frac{\ell}{2} \leq m, n \leq \frac{\ell}{2}$. So we can write

$$\phi \;=\; \sum_{\ell,m,n} \phi_{\ell mn} Y_\ell^{mn}\ , \tag{3.164}$$

[21] Outside this range there are complications, see [106], for example, the cancellation previously discussed for long multiplets would fail.

and similarly for the auxiliary field F. The fermionic field ψ_α can be expanded in spinorial harmonics.

Integrating over S^3 and using the orthonormality of the spherical harmonics, the action of a four-dimensional chiral multiplet gives rise to a one-dimensional action for an infinite number of fields. These arrange in multiplets of supersymmetric quantum mechanics labeled by ℓ, m, n, and one can check that the Lagrangian of each of these multiplets takes the form (3.131). Here we do not need to present all details of the reduction. All we need to know is how the R-charge r and the quantum numbers ℓ, m, n map into the parameters σ, p, ν entering in (3.131) and characterize each multiplet in the supersymmetric quantum mechanics. Actually, the discussion in subsection 3.5.3 shows that for the purpose of determining the vacuum energy we just need to know when a multiplet is shortened (namely when $p = 0$), if it is a chiral or a Fermi multiplet, and what is the value of its charge σ.

By comparing the four-dimensional algebra (3.126) with (3.128), we deduce that we must identify (restoring the S^3 radius r_3) $\Sigma = \frac{1}{r_3}(R + 2J_l^3)$, and therefore

$$\sigma = \frac{1}{r_3}(r + 2m) . \tag{3.165}$$

Moreover, reducing the four-dimensional Lagrangian to one dimension, one finds[22]

$$\begin{aligned} p^2 &= \frac{1}{r_3^2}(\ell - 2m)(\ell + 2 + 2m) , \\ \nu &= -\frac{1}{r_3}(2m + 1) , \end{aligned} \tag{3.166}$$

hence the shortening condition $p = 0$ is satisfied if and only if $m = \ell/2$ or $m = -\ell/2 - 1$. In the former case a chiral multiplet is obtained with charge $\sigma = \frac{1}{r_3}(\ell + r)$. In the latter case a Fermi multiplet is obtained with charge $\sigma = -\frac{1}{r_3}(\ell + 2 - r)$. Recalling (3.159), (3.163) we conclude that the respective contribution to the vacuum energy is:

$$\begin{aligned} &\text{chiral} \quad \left(m = \tfrac{\ell}{2}\right) : && \langle H_{\text{chiral}} \rangle = \tfrac{1}{2r_3}(\ell + r) , \\ &\text{Fermi} \quad (m = -\tfrac{\ell}{2} - 1) : && \langle H_{\text{Fermi}} \rangle = -\tfrac{1}{2r_3}(\ell + 2 - r) . \end{aligned} \tag{3.167}$$

The expectation value of the Hamiltonian is obtained by adding up the contributions

[22] More generally, one could easily restore the dependence on the parameter κ. This affects only ν but not p^2 and σ. In the notation of section 3.2, one finds that the parameter ν is related to the parameters in the four-dimensional Lagrangian as $r_3\nu = -2m - \frac{3}{2}r - \kappa(\frac{3}{2}r - \epsilon)$.

of all chiral and Fermi multiplets:

$$\begin{aligned}\langle H_{\text{susy}}\rangle &= \sum_{\text{chiral}}\langle H_{\text{chiral}}\rangle + \sum_{\text{Fermi}}\langle H_{\text{Fermi}}\rangle \\ &= \sum_{\ell\geq 0}\frac{1}{2r_3}(\ell+1)(\ell+r) - \sum_{\ell\geq 0}\frac{1}{2r_3}(\ell+1)(\ell+2-r)\ , \end{aligned} \tag{3.168}$$

where the $(\ell+1)$ factor comes from the degeneracy associated with $SU(2)_r$.

To regularize the sum, we dress the terms in the sum with some decreasing weights. To do this in a supersymmetric fashion, we can decompose H as a sum of Hamiltonians acting on the Hilbert space of a single free $1d$ multiplet

$$H_{\text{susy}} = \sum_{\ell,m,n} H_{\ell,m,n}\ , \tag{3.169}$$

and regularize the sum with a function of the $H_{\ell,m,n}$ operators, for instance

$$H_{\text{susy}} = \sum_{\ell,m,n} H_{\ell,m,n}\mathrm{e}^{-2\,t\,r_3|H_{\ell,m,n}|}\ , \tag{3.170}$$

with t a positive number. This yields

$$\langle H_{\text{susy}}\rangle = \sum_{\ell\geq 0}\frac{1}{2r_3}(\ell+1)(\ell+r)\mathrm{e}^{-t(\ell+r)} - \sum_{\ell\geq 0}\frac{1}{2r_3}(\ell+1)(\ell+2-r)\mathrm{e}^{-t(\ell+2-r)}\ . \tag{3.171}$$

Taking the small t limit and dropping the diverging term in t^{-2},[23] we obtain a regularized result for the vacuum energy,

$$E_{\text{susy}} = \langle H_{\text{susy}}\rangle = \frac{4}{27r_3}(\mathbf{a}+3\mathbf{c})\ , \tag{3.172}$$

in agreement with the result (3.51) obtain above using the Hurwitz zeta function.

One could consider a supersymmetric regularization with a different function $f(tH_{\ell,m,n})$ of the $H_{\ell,m,n}$ operators. It can be shown, using an Euler-MacLaurin expansion that for all smooth functions f such that $f(0)=1$ (and such that the series converges), one obtains the same result for the finite piece in the small t expansion (see appendix C of [3] for a related application). This is in agreement with the fact that the supersymmetric Casimir energy is unambiguous.

It is possible to contrast our results with several previous works in which localization techniques on $S^1\times S^3$ were utilized. Comparing with [26] (see also [97] and [107] where similar localization techniques are used in other topologies), one finds agreement regarding the vacuum energy. However, as was discussed in the appendix of [3], the regularization scheme of [26] in fact does not preserve supersymmetry, as

[23] The diverging term can be associated to the four-dimensional Einstein-Hilbert counterterm.

it violates certain SUSY Ward identities in the small circle limit.

3.6 Conclusions

In this chapter, we studied the supersymmetric Casimir energy of $\mathcal{N} = 1$ theories on the background $\mathbb{R} \times S^3$, as introduced in [26], in the simplest case where the partition function only depends on one fugacity.

Firstly, by revisiting the localization computation in [26], we have verified explicitly that its value does not depend on the choice of the parameter κ, characterizing the background fields A and V, as expected. Secondly, we reproduced it by evaluating the expectation value of the BPS Hamiltonian that appears in the definition of the supersymmetric index, as anticipated in [108]. Our computations also clarify the relation of the supersymmetric Casimir energy with the Casimir energy of free conformal fields theories, demonstrating that these two quantities arise as the expectation values of two *different* Hamiltonians, evaluated using the *same* zeta function regularization method.

We then showed in section 3.5, that in fact the supersymmetric Casimir energy is free of ambiguities, provided the chosen regularization scheme is compatible with supersymmetry. We considered in this chapter only the case $\mathbb{R} \times S^3$ (and $S^1 \times S^3$) where the three-sphere is round. In fact, [3] further included a proof that the supersymmetric Casimir energy can be computed unambiguously on a deformed three-sphere. In this case, the explicit spherical harmonics and eigenvalues are not available, however, due to the shortening condition analogous to (3.157), the full reduction on the three-sphere is not needed. The result obtained in this generalized setting is shown to be consistent with the computations of this chapter on the round sphere.

In the next chapter we address the issue of the holographic dual gravity description of the theories considered above on the round $\mathbb{R} \times S^3$.

Chapter 4

Supersymmetric solutions of five-dimensional minimal gauged supergravity

In this chapter, we study supersymmetric solutions of five-dimensional minimal gauged supergravity. We begin in section 4.1 by reviewing the formalism of [32], where all supersymmetric solutions of this theory were classified. We will focus here on the timelike class. Next, in section 4.2, we start from an ansatz with an *orthotoric* Kähler base metric, finding a five-dimensional AlAdS_5 solution comprising five non-trivial parameters. When two of the parameters are set to zero, the solution is AAdS_5 and is related to that of [76] by a change of coordinates. For specific values of the parameters of [76] this change of coordinates becomes singular. We interpret this in section 4.2.5 in terms of a scaling limit of the orthotoric ansatz, leading to certain non-orthotoric Kähler metrics previously employed in the search for supergravity solutions. This proves that our orthotoric ansatz, together with its scaling limits encompasses *all* known supersymmetric solutions to minimal gauged supergravity in the timelike class.

In section 4.3 we focus on certain non-trivial geometries with no horizon contained in the solution of [76], called "topological solitons". These are *a priori* natural candidates to describe pure states of an $\mathcal{N}=1$ SCFTs. We investigate whether among these solutions we can match holographically the vacuum state of an $\mathcal{N}=1$ SCFT on the cylinder $\mathbb{R}\times S^3$, and in particular the non-vanishing supersymmetric vacuum expectation values of the energy and R-charge, as presented in chapter 3. Some basic requirements following from the supersymmetry algebra lead us to consider a 1/2 BPS topological soliton presented in [109]. Although a direct comparison of the charges shows that this fails to describe the vacuum state of the dual SCFT, in the process we clarify some aspects of these topological solitons. In section 4.4 we make some concluding remarks on this chapter. Appendix D includes a proof of the uniqueness of a supersymmetric solution of minimal gauged supergravity with

$SO(4)$ symmetry.

This chapter is based on [4], and we work in $(-++++)$ signature.

4.1 Supersymmetric solutions from Kähler bases

In this section we briefly review the conditions for bosonic supersymmetric solutions to five-dimensional minimal gauged supergravity found in [32], focusing on the timelike class. The bosonic action of minimal gauged supergravity is

$$S \;=\; \frac{1}{2\kappa_5^2}\int\left[(R_5+12g^2)\ast_5 1-\frac{1}{2}F\wedge\ast_5 F+\frac{1}{3\sqrt{3}}A\wedge F\wedge F\right]\,, \tag{4.1}$$

where R_5 is the Ricci scalar of the five-dimensional metric $g_{\mu\nu}$, A is the graviphoton $U(1)$ gauge field, $F=\mathrm{d}A$ is its field strength, $g>0$ parametrizes the cosmological constant, and G_5 is Newton's constant. The Einstein and Maxwell equations of motion are

$$\begin{aligned} R^{(5)}_{\mu\nu}+4g^2g_{\mu\nu}-\frac{1}{2}F_{\mu\kappa}F_{\nu}{}^{\kappa}+\frac{1}{12}g_{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda} &= 0\,,\\ \mathrm{d}\ast_5 F-\frac{1}{\sqrt{3}}F\wedge F &= 0\,. \end{aligned} \tag{4.2}$$

A bosonic background is supersymmetric if there is a non-zero Dirac spinor ϵ satisfying

$$\left[\nabla^{(5)}_{\mu}-\frac{\mathrm{i}}{8\sqrt{3}}\left(\Gamma_{\mu}{}^{\nu\kappa}-4\delta^{\nu}_{\mu}\Gamma^{\kappa}\right)F_{\nu\kappa}-\frac{g}{2}\big(\Gamma_{\mu}+\sqrt{3}\,\mathrm{i}A_{\mu}\big)\right]\epsilon \;=\; 0\,, \tag{4.3}$$

where the gamma-matrices obey the Clifford algebra $\{\Gamma_\mu,\Gamma_\nu\}=2g_{\mu\nu}$. By assuming the existence of such a Killing spinor ϵ, the authors of [32] showed that all such solutions admit a Killing vector $\mathcal{V}$ constructed as a bilinear in ϵ that is either timelike or null. Here we will discuss the timelike class.

By choosing coordinates in which $\mathcal{V}=\partial/\partial t$, the five-dimensional metric can be put in the form

$$\mathrm{d}s^2_5 \;=\; -f^2\,(\mathrm{d}t+\omega)^2+f^{-1}\,\mathrm{d}s^2_B\,, \tag{4.4}$$

where $\mathrm{d}s^2_B$ denotes the metric on a four-dimensional base B transverse to $\mathcal{V}$, while f and ω are a positive function and a one-form on B, respectively. Supersymmetry requires B to be Kähler. This means that B admits a real non-degenerate two-form X^1 that is closed, *i.e.* $\mathrm{d}X^1=0$, and such that $X^1{}_m{}^n$ is an integrable complex structure (m,n denote curved indices on B, and we raise the index of X^1_{mn} with the inverse metric on B). It will be useful to recall that a four-dimensional Kähler manifold also admits a complex two-form Ω of type $(2,0)$ satisfying

$$\nabla_m \Omega_{np} + \mathrm{i}\mathcal{P}_m \Omega_{np} \;=\; 0 \;, \tag{4.5}$$

where $\mathcal{P}$ is a potential for the Ricci form, *i.e.* $\mathcal{R} = \mathrm{d}\mathcal{P}$. The Ricci form is a closed two-form defined as $\mathcal{R}_{mn} = \frac{1}{2} R_{mnpq} (X^1)^{pq}$, where R_{mnpq} is the Riemann tensor on B. Moreover, splitting $\Omega = X^2 + \mathrm{i}X^3$, the triple of real two-forms X^I, $I = 1, 2, 3$, satisfies the quaternion algebra:

$$X^I{}_m{}^p X^J{}_p{}^n \;=\; -\delta^{IJ} \delta_m{}^n + \epsilon^{IJK} X^K{}_m{}^n \;. \tag{4.6}$$

We choose the orientation on B by fixing the volume form as $\mathrm{vol}_B = -\frac{1}{2} X^1 \wedge X^1$. It follows that the X^I are a basis of anti-self-dual forms on B, *i.e.* $*_B X^I = -X^I$.

The geometry of the Kähler base B determines the whole solution, namely f and ω in the five-dimensional metric (4.4), and the graviphoton field strength F. The function f is fixed by supersymmetry as

$$f \;=\; -\frac{24 g^2}{R} \;, \tag{4.7}$$

where R is the Ricci scalar of $\mathrm{d}s^2_B$; this is required to be everywhere non-zero.

The expression for the Maxwell field strength is

$$F \;=\; -\sqrt{3}\, \mathrm{d}\Big[f (\mathrm{d}t + \omega) + \frac{1}{3g} \mathcal{P} \Big] \;. \tag{4.8}$$

Note that the Killing vector $\mathcal{V}$ also preserves F, hence it is a symmetry of the solution.

It remains to compute the one-form ω. This is done by solving the equation

$$\mathrm{d}\omega \;=\; f^{-1} (G^+ + G^-) \;, \tag{4.9}$$

where the two-forms $G^\pm$, satisfying the (anti)-self-duality relations $*_B G^\pm = \pm G^\pm$, are determined as follows. Supersymmetry states that G^+ be given as

$$G^+ \;=\; -\frac{1}{2g} \Big(\mathcal{R} - \frac{R}{4} X^1 \Big) \;. \tag{4.10}$$

Expanding G^- in the basis of anti-self-dual two-forms as[1]

$$G^- \;=\; \frac{1}{2gR} (\lambda^1 X^1 + \lambda^2 X^2 + \lambda^3 X^3) \;, \tag{4.11}$$

[1]Our λ^I are rescaled by a factor of $2gR$ compared to those in [32].

one finds that the Maxwell equation fixes λ^1 as

$$\lambda^1 \;=\; \frac{1}{2}\nabla^2 R + \frac{2}{3}R_{mn}R^{mn} - \frac{1}{3}R^2 \;, \tag{4.12}$$

where ∇^2 and R_{mn} are the Laplacian and Ricci tensor on B, respectively. The remaining two components, λ^2, λ^3, only have to be compatible with the requirement that the right hand side of (4.9) be closed,

$$\mathrm{d}\left[f^{-1}(G^+ + G^-)\right] \;=\; 0 \;. \tag{4.13}$$

Inserting (4.7), (4.10), and (4.12) into (4.13) and taking the Hodge dual, one arrives at the equation

$$\mathrm{Im}\left[\,\overline{\Omega}_m{}^n(\partial_n + \mathrm{i}\mathcal{P}_n)(\lambda^2 + \mathrm{i}\lambda^3)\right] + \Xi_m \;=\; 0 \;, \tag{4.14}$$

where

$$\Xi_m \;=\; R_{mn}\partial^n R + \partial_m\left(\frac{1}{2}\nabla^2 R + \frac{2}{3}R_{pq}R^{pq} - \frac{1}{3}R^2\right) \;. \tag{4.15}$$

Acting on (4.14) with $\Pi_p{}^q X^3{}_q{}^m$, where $\Pi = \frac{1}{2}\left(\mathbb{1} + \mathrm{i}X^1\right)$ is the projector on the $(1,0)$ part, one obtains the equivalent form

$$D^{(1,0)}(\lambda^2 + \mathrm{i}\lambda^3) + \Theta^{(1,0)} \;=\; 0 \;, \tag{4.16}$$

where $D^{(1,0)}_m = \Pi_m{}^n(\nabla_n + \mathrm{i}\mathcal{P}_n)$ is the holomorphic Kähler covariant derivative, and we defined $\Theta^{(1,0)}_m = \Pi_m{}^n X^3{}_n{}^p \Xi_p$. Equation (4.16) determines $\lambda^2 + \mathrm{i}\lambda^3$, and hence G^-, up to an anti-holomorphic function. This concludes the analysis of the timelike case as presented in [32].

It was first pointed out in [79] that for equation (4.13) to admit a solution, a constraint on the Kähler geometry must be satisfied. Hence not all four-dimensional Kähler bases give rise to supersymmetric solutions. While in [79] this was shown for a specific family of Kähler bases, here we provide a general formulation as it first appeared in ref. [4]. Taking the divergence of (4.14) and using (4.5) we find

$$\nabla^m \Xi_m \;=\; 0 \;, \tag{4.17}$$

that is

$$\nabla^2\left(\frac{1}{2}\nabla^2 R + \frac{2}{3}R_{pq}R^{pq} - \frac{1}{3}R^2\right) + \nabla^m(R_{mn}\partial^n R) \;=\; 0 \;. \tag{4.18}$$

We thus obtain a rather complicated sixth-order equation constraining the Kähler metric.[2] We observe that the term $(\nabla^2)^2 R + 2\nabla^m(R_{mn}\partial^n R)$ corresponds to the real part of the Lichnerowicz operator acting on R, which vanishes for *extremal* Kähler

[2]It can also be derived starting from the observation that since $D^{(1,0)}$ is a good differential, namely $\left(D^{(1,0)}\right)^2 = 0$, equation (4.16) has the integrability condition $D^{(1,0)}\Theta^{(1,0)} = 0$. The latter is an a priori complex equation, however one finds that the real part is automatically satisfied while the imaginary part is equivalent to (4.18).

metrics (see *e.g.* [110, sect.4.1]). Thus in this case (4.18) reduces to $\nabla^2\left(2R_{pq}R^{pq}-R^2\right)=0$. If the Kähler metric has constant Ricci scalar, the constraint simplifies further to $\nabla^2(R_{pq}R^{pq})=0$. Finally, if the Kähler metric is homogeneous, or Einstein, then $\Xi=0$ and the constraint is trivially satisfied.

To summarize, the five-dimensional metric and the gauge field strength are determined by the four-dimensional Kähler geometry up to an anti-holomorphic function. The Kähler metric is constrained by the sixth-order equation (4.18). Moreover, one needs $R\neq 0$. The conditions spelled out above are necessary and sufficient for obtaining a supersymmetric solution of the timelike class. The solutions preserve at least 1/4 of the supersymmetry, namely two real supercharges.

4.2 Orthotoric Kähler basis

4.2.1 The ansatz

In this section we construct supersymmetric solutions following the procedure described above. We start from a very general ansatz for the four-dimensional base, given by a class of local Kähler metrics known as *orthotoric*. These were introduced in ref. [111], to which we refer for an account of their mathematical properties.[3]

The orthotoric Kähler metric with toric Killing vectors $\partial/\partial\Phi$, $\partial/\partial\Psi$ reads

$$g^2\,\mathrm{d}s^2_B \;=\; \frac{\eta-\xi}{\mathcal{F}(\xi)}\mathrm{d}\xi^2+\frac{\mathcal{F}(\xi)}{\eta-\xi}(\mathrm{d}\Phi+\eta\mathrm{d}\Psi)^2+\frac{\eta-\xi}{\mathcal{G}(\eta)}\mathrm{d}\eta^2+\frac{\mathcal{G}(\eta)}{\eta-\xi}(\mathrm{d}\Phi+\xi\mathrm{d}\Psi)^2\;, \tag{4.19}$$

where $\mathcal{F}(\xi)$ and $\mathcal{G}(\eta)$ are a priori arbitrary functions. The Kähler form has a universal expression, independent of $\mathcal{F}(\xi)$, $\mathcal{G}(\eta)$:

$$X^1 \;=\; \frac{1}{g^2}\,\mathrm{d}\left[(\eta+\xi)\mathrm{d}\Phi+\eta\xi\,\mathrm{d}\Psi\right]\;. \tag{4.20}$$

The term orthotoric means that the momentum maps $\eta+\xi$ and $\eta\xi$ for the Hamiltonian Killing vector fields $\partial/\partial\Phi$ and $\partial/\partial\Psi$, respectively, have the property that the one-forms $\mathrm{d}\xi$, $\mathrm{d}\eta$ are orthogonal. As a consequence, the Kähler metric does not contain a $\mathrm{d}\eta\mathrm{d}\xi$ term.

[3] This ansatz was also considered in [112], however only the case $\mathcal{F}(x)=-\mathcal{G}(x)$, where these are cubic polynomials, was discussed there. In this case the metric (4.19) is equivalent to the Bergmann metric on $SU(2,1)/S(U(2)\times U(1))$. Orthotoric metrics also appear in Sasaki-Einstein geometry: as shown in [113], the Kähler-Einstein bases of $L^{p,q,r}$ Sasaki-Einstein manifolds [114] are of this type.

It is convenient to introduce an orthonormal frame

$$E_1 = \frac{1}{g}\sqrt{\frac{\eta-\xi}{\mathcal{F}(\xi)}}\,\mathrm{d}\xi\ , \qquad E_2 = \frac{1}{g}\sqrt{\frac{\mathcal{F}(\xi)}{\eta-\xi}}\,(\mathrm{d}\Phi+\eta\mathrm{d}\Psi)\ ,$$
$$E_3 = \frac{1}{g}\sqrt{\frac{\eta-\xi}{\mathcal{G}(\eta)}}\,\mathrm{d}\eta\ , \qquad E_4 = \frac{1}{g}\sqrt{\frac{\mathcal{G}(\eta)}{\eta-\xi}}\,(\mathrm{d}\Phi+\xi\mathrm{d}\Psi)\ , \tag{4.21}$$

with volume form $\mathrm{vol}_B = -E_1\wedge E_2\wedge E_3\wedge E_4$. Then the Kähler form can be written as

$$X^1 \;=\; E_1\wedge E_2 + E_3\wedge E_4\ . \tag{4.22}$$

For the complex two-form Ω we can take

$$\Omega \;=\; X^2 + \mathrm{i}X^3 \;=\; (E_1 - \mathrm{i}E_2)\wedge(E_3-\mathrm{i}E_4)\ . \tag{4.23}$$

This satisfies the properties (4.5), (4.6), with the Ricci form potential given by

$$\mathcal{P} \;=\; \frac{\mathcal{F}'(\xi)(\mathrm{d}\Phi+\eta\mathrm{d}\Psi)+\mathcal{G}'(\eta)(\mathrm{d}\Phi+\xi\mathrm{d}\Psi)}{2(\xi-\eta)}\ . \tag{4.24}$$

Other formulae that we will need are the Ricci scalar

$$R \;=\; g^2\,\frac{\mathcal{F}''(\xi)+\mathcal{G}''(\eta)}{\xi-\eta}\ , \tag{4.25}$$

and its Laplacian

$$\nabla^2 R \;=\; \frac{g^2}{\eta-\xi}\left[\partial_\xi(\mathcal{F}\,\partial_\xi R)+\partial_\eta(\mathcal{G}\,\partial_\eta R)\right]\ . \tag{4.26}$$

4.2.2 The solution

To construct the solution we insert our orthotoric ansatz into the supersymmetry equations of section 4.1. Equation (4.7) gives for the function f,

$$f \;=\; \frac{24(\eta-\xi)}{\mathcal{F}''(\xi)+\mathcal{G}''(\eta)}\ . \tag{4.27}$$

In order to solve eq. (4.9) for ω, we need to first construct G^+ and G^-. From eq. (4.10) we obtain

$$G^+ \;=\; \frac{1}{8g}(\partial_\xi\mathcal{H}-\partial_\eta\mathcal{H})\,(E_1\wedge E_2 - E_3\wedge E_4)\ , \tag{4.28}$$

where we introduced the useful combination

$$\mathcal{H}(\eta,\xi) \;=\; g^2\,\frac{\mathcal{F}'(\xi)+\mathcal{G}'(\eta)}{\eta-\xi}\ . \tag{4.29}$$

We recall that $G^- = \frac{1}{2gR}\sum_{I=1}^3 \lambda^I X^I$, and we have to compute the functions $\lambda^1, \lambda^2, \lambda^3$. Equation (4.12) gives

$$\lambda^1 \;=\; \frac{1}{2}\nabla^2 R - \frac{2}{3}\,\partial_\xi\mathcal{H}\,\partial_\eta\mathcal{H}\;, \tag{4.30}$$

where $\nabla^2 R$ was expressed in terms of orthotoric data above. In order to solve for λ^2, λ^3, we have to analyze the constraint (4.18) on the Kähler metric. Plugging our ansatz in, we obtain the equation

$$\begin{aligned}&\partial_\xi\left[\mathcal{F}\,\partial_\xi\mathcal{H}\,\partial_\xi(\partial_\xi\mathcal{H}+\partial_\eta\mathcal{H})+\mathcal{F}\,\partial_\xi\left(\nabla^2R-\tfrac{4}{3}\,\partial_\xi\mathcal{H}\,\partial_\eta\mathcal{H}\right)\right]\\ +\;&\partial_\eta\left[\mathcal{G}\,\partial_\eta\mathcal{H}\,\partial_\eta(\partial_\xi\mathcal{H}+\partial_\eta\mathcal{H})+\mathcal{G}\,\partial_\eta\left(\nabla^2R-\tfrac{4}{3}\,\partial_\xi\mathcal{H}\,\partial_\eta\mathcal{H}\right)\right] \;=\; 0\;.\end{aligned} \tag{4.31}$$

This is a complicated sixth-order equation for the two functions $\mathcal{F}(\xi)$ and $\mathcal{G}(\eta)$. In ref. [4] the general solution to equation (4.31) was not found, however, a cubic polynomial solution was presented,

$$\begin{aligned}\mathcal{G}(\eta) &= g_4(\eta-g_1)(\eta-g_2)(\eta-g_3)\;,\\ \mathcal{F}(\xi) &= -\mathcal{G}(\xi)+f_1(\xi+f_0)^3\;,\end{aligned} \tag{4.32}$$

comprising six arbitrary[4] parameters $g_1,\dots,g_4,\ f_0,\ f_1$. We thus continue assuming that $\mathcal{F}$ and $\mathcal{G}$ take the form (4.32). We can then solve eq. (4.14) for λ^2, λ^3. Assuming a dependence on η, ξ only, the solution is

$$\lambda^2 + i\lambda^3 \;=\; i\,g^4\,\frac{\mathcal{F}'''+\mathcal{G}'''}{(\eta-\xi)^3}\sqrt{\mathcal{F}(\xi)\mathcal{G}(\eta)}\;+g^4\frac{c_2+ic_3}{\sqrt{\mathcal{F}(\xi)\mathcal{G}(\eta)}}\;, \tag{4.33}$$

with c_2, c_3 real integration constants. One can promote c_2+ic_3 to an arbitrary anti-holomorphic function, however we will not discuss such generalization here (see [79] for an example where this has been done explicitly).

We now have all the ingredients to solve eq. (4.9) and determine ω. The solution is

$$\begin{aligned}\omega \;=\;& \frac{\mathcal{F}'''+\mathcal{G}'''}{48g(\eta-\xi)^2}\Big\{\left[\mathcal{F}(\xi)+(\eta-\xi)\left(\tfrac{1}{2}\mathcal{F}'(\xi)-\tfrac{1}{4}\mathcal{F}'''(\xi)(f_0+\xi)^2\right)\right](\mathrm{d}\Phi+\eta\mathrm{d}\Psi)\\ &+\mathcal{G}(\eta)(\mathrm{d}\Phi+\xi\mathrm{d}\Psi)\Big\} - \frac{\mathcal{F}'''\mathcal{G}'''}{288g}\left[(\eta+\xi)\mathrm{d}\Phi+\eta\xi\,\mathrm{d}\Psi\right]\\ &-\frac{c_2}{48g}\Big(I_1\frac{\xi\mathrm{d}\xi}{\mathcal{F}(\xi)}+I_2\frac{\eta\mathrm{d}\eta}{\mathcal{G}(\eta)}+\Phi\mathrm{d}\Psi\Big)\\ &-\frac{c_3}{48g}\left[(I_1-I_2)\mathrm{d}\Phi+(I_3-I_4)\mathrm{d}\Psi\right]+\mathrm{d}\mathcal{X}\;,\end{aligned} \tag{4.34}$$

[4]This includes the case where $g_4\to 0$ and one or more roots diverge, so that the cubic $\mathcal{G}$ degenerates to a polynomial of lower degree. Similarly for $\mathcal{F}$.

where

$$I_1 = \int \frac{\mathrm{d}\eta}{\mathcal{G}(\eta)} \,, \qquad I_2 = \int \frac{\mathrm{d}\xi}{\mathcal{F}(\xi)} \,, \qquad I_3 = \int \frac{\eta\,\mathrm{d}\eta}{\mathcal{G}(\eta)} \,, \qquad I_4 = \int \frac{\xi\,\mathrm{d}\xi}{\mathcal{F}(\xi)} \,. \tag{4.35}$$

Moreover, $\mathrm{d}\mathcal{X}$ is an arbitrary locally exact one-form. In the five-dimensional metric this can be reabsorbed by a change of the t coordinate. For $\mathcal{F}$ and $\mathcal{G}$ as in (4.32), the integrals $I_1, \ldots, I_4$ can be expressed in terms of the roots of the polynomials. We have:

$$I_1 = \frac{\log(\eta - g_1)}{g_4(g_1 - g_2)(g_1 - g_3)} + \mathrm{cycl}(1,2,3) \,, \qquad I_3 = \frac{g_1 \log(\eta - g_1)}{g_4(g_1 - g_2)(g_1 - g_3)} + \mathrm{cycl}(1,2,3) \,, \tag{4.36}$$

and similarly for I_2 and I_4 (although the roots of $\mathcal{F}$ in (4.32) expressed in terms of the parameters $g_1, \ldots, g_4$, f_0, f_1 are less simple). Here, $\mathrm{cycl}(1,2,3)$ denotes cyclic permutations of the roots.

Note that if $c_2 \neq 0$ then ω explicitly depends on one of the angular coordinates Φ, Ψ, hence the $U(1) \times U(1)$ symmetry of the orthotoric base is broken to a single $U(1)$ in the five-dimensional metric.

To summarize, we started from the orthotoric ansatz (4.19) for the four-dimensional Kähler metric, studied the sixth-order constraint (4.18) and found a solution in terms of cubic polynomials $\mathcal{F}$, $\mathcal{G}$ containing six arbitrary parameters, *cf.* (4.32). We also provided explicit expressions for $\mathcal{P}$, f and ω (*cf.* (4.24), (4.27), (4.34)), with the solution for ω containing the additional parameters c_2, c_3. Inserting these expressions in the metric (4.4) and Maxwell field (4.8), we thus obtain a supersymmetric solution to minimal gauged supergravity controlled by eight parameters. We now show that three of the six parameters in the polynomials are actually trivial in the five-dimensional solution.

4.2.3 Triviality of three parameters

As a first thing, we observe that one is always free to rescale the four-dimensional Kähler base by a constant factor. This is because the spinor solving the supersymmetry equation (4.3) is defined up to a multiplicative constant, and the spinor bilinears inherit such rescaling freedom. This leads to the transformation

$$\begin{aligned} &X^I \to \varepsilon\, X^I \,, \qquad f \to \varepsilon\, f \,, \qquad t \to \varepsilon^{-1} t \,, \\ &\mathrm{d}s^2_B \to \varepsilon\, \mathrm{d}s^2_B \,, \qquad \mathcal{P} \to \mathcal{P} \,, \qquad \omega \to \varepsilon^{-1}\omega \,, \end{aligned} \tag{4.37}$$

where ε is a non-zero constant. Clearly this leaves the five-dimensional metric (4.4) and the gauge field (4.8) invariant.

Let us now consider a supersymmetric solution whose Kähler base metric $\mathrm{d}s^2_B$ is in the orthotoric form (4.19), with some given functions $\mathcal{F}(\xi)$ and $\mathcal{G}(\eta)$. Then we can use the symmetry above to rescale these two functions. Indeed after performing the transformation we have $(\mathrm{d}s_B^2)^{\text{old}} = \varepsilon\,(\mathrm{d}s_B^2)^{\text{new}}$, and the new Kähler metric is again in orthotoric form, with the redefinitions

$$\mathcal{F}^{\text{old}} = \varepsilon^{-1}\mathcal{F}^{\text{new}}\ , \qquad \mathcal{G}^{\text{old}} = \varepsilon^{-1}\mathcal{G}^{\text{new}}\ , \qquad \Phi^{\text{old}} = \varepsilon\,\Phi^{\text{new}}\ , \qquad \Psi^{\text{old}} = \varepsilon\,\Psi^{\text{new}}\ . \tag{4.38}$$

Hence the overall scale of $\mathcal{F}$ and $\mathcal{G}$ is irrelevant as far as the five-dimensional solution is concerned. A slightly more complicated transformation that we can perform is

$$\begin{aligned}
\xi^{\text{old}} &= \varepsilon_2\xi^{\text{new}} + \varepsilon_3\ , & \eta^{\text{old}} &= \varepsilon_2\eta^{\text{new}} + \varepsilon_3\ ,\\
\Psi^{\text{old}} &= \varepsilon_1\varepsilon_2\Psi^{\text{new}}\ , & \Phi^{\text{old}} &= \varepsilon_1(\varepsilon_2^2\Phi^{\text{new}} - \varepsilon_2\varepsilon_3\Psi^{\text{new}})\ ,\\
\mathcal{F}^{\text{old}}(\xi^{\text{old}}) &= \varepsilon_1^{-1}\mathcal{F}^{\text{new}}(\xi^{\text{new}})\ , & \mathcal{G}^{\text{old}}(\eta^{\text{old}}) &= \varepsilon_1^{-1}\mathcal{G}^{\text{new}}(\eta^{\text{new}})\ .
\end{aligned} \tag{4.39}$$

with arbitrary constants $\varepsilon_1 \neq 0$, $\varepsilon_2 \neq 0$ and ε_3, such that $\varepsilon_1\varepsilon_2^3 = \varepsilon$. It is easy to see that the new metric $(\mathrm{d}s_B^2)^{\text{new}}$ is again orthotoric, though with different cubic functions $\mathcal{F}$ and $\mathcal{G}$ compared to the old ones.

We conclude that a supersymmetric solution with orthotoric Kähler base is locally equivalent to another orthotoric solution, with functions

$$\mathcal{F}^{\text{new}}(\xi) = \varepsilon_1\mathcal{F}^{\text{old}}(\varepsilon_2\xi + \varepsilon_3)\ , \qquad \mathcal{G}^{\text{new}}(\eta) = \varepsilon_1\mathcal{G}^{\text{old}}(\varepsilon_2\eta + \varepsilon_3)\ . \tag{4.40}$$

Using this freedom, we can argue that three of the six parameters in our orthotoric solution are trivial. In the next section we will show that the remaining ones are not trivial by relating our solution with $c_2 = c_3 = 0$ to the solution of [76].

4.2.4 Relation to [76]

The authors of [76] provide a four-parameter family of AAdS solutions to minimal five-dimensional gauged supergravity. The generic solution preserves $U(1)\times U(1)\times\mathbb{R}$ symmetry (where $\mathbb{R}$ is the time direction) and is non-supersymmetric. By fixing one of the parameters, one obtains a family of supersymmetric solutions, controlled by the three remaining parameters a, b, m. This includes the most general supersymmetric black hole free of closed timelike curves (CTC's) known in minimal gauged supergravity, as well as a family of topological solitons. Generically, the supersymmetric solutions are 1/4 BPS in the five-dimensional theory, namely they preserve two real supercharges. For $b = a$ or $b = -a$, the symmetry is enhanced to $SU(2) \times U(1) \times \mathbb{R}$.

We find that upon a change of coordinates the supersymmetric solution of [76] fits in our orthotoric solution, with polynomial functions $\mathcal{F}$, $\mathcal{G}$ of the type discussed above. In detail, the five-dimensional metric and gauge field strength of [76] match (4.4) and (4.8), with the data given in the previous section and $c_2 = c_3 = 0$.

The change of coordinates is

$$\begin{aligned}
t_{\text{CCLP}} &= t \\
\theta_{\text{CCLP}} &= \frac{1}{2}\arccos\eta \\
r^2_{\text{CCLP}} &= \frac{1}{2}(a^2-b^2)\tilde{m}\,\xi + \frac{1}{g}\left[(a+b)\tilde{m} + a + b + abg\right] + \frac{1}{2}(a+b)^2\tilde{m}\ , \\
\phi_{\text{CCLP}} &= g\,t - 4\frac{1-a^2g^2}{(a^2-b^2)g^2\tilde{m}}\left(\Phi - \Psi\right)\ , \\
\psi_{\text{CCLP}} &= g\,t - 4\frac{1-b^2g^2}{(a^2-b^2)g^2\tilde{m}}\left(\Phi + \Psi\right)\ ,
\end{aligned} \tag{4.41}$$

where "CCLP" labels the coordinates of [76]. Here, we found convenient to trade m for

$$\tilde{m} = \frac{m\,g}{(a+b)(1+ag)(1+bg)(1+ag+bg)} - 1\,, \tag{4.42}$$

which is defined so that the black hole solution of [76] corresponds to $\tilde{m}=0$. The cubic polynomials $\mathcal{F}(\xi)$ and $\mathcal{G}(\eta)$ read

$$\begin{aligned}
\mathcal{G}(\eta) &= -\frac{4}{(a^2-b^2)g^2\tilde{m}}(1-\eta^2)\left[(1-a^2g^2)(1+\eta)+(1-b^2g^2)(1-\eta)\right]\ , \\
\mathcal{F}(\xi) &= -\mathcal{G}(\xi) - 4\,\frac{1+\tilde{m}}{\tilde{m}}\left(\frac{2+ag+bg}{(a-b)g}+\xi\right)^3\ ,
\end{aligned} \tag{4.43}$$

and are clearly of the form (4.32).[5] The function $\mathcal{X}$ in (4.34) is $\mathcal{X} = -\frac{2\Psi}{g\tilde{m}}$. The Killing vector arising as a bilinear of the spinor ϵ solving the supersymmetry equation (4.3) is

$$V = \frac{\partial}{\partial t} = \frac{\partial}{\partial t_{\text{CCLP}}} + g\frac{\partial}{\partial\phi_{\text{CCLP}}} + g\frac{\partial}{\partial\psi_{\text{CCLP}}}\ . \tag{4.44}$$

We conclude that for $c_2 = c_3 = 0$, the family of supersymmetric solutions we have constructed is (at least locally) equivalent to the supersymmetric solutions of [76].

When either c_2 or c_3 (or both) are switched on, the boundary metric is no more conformally flat, hence the solution becomes AlAdS$_5$ and is not diffeomorphic to the $c_2 = c_3 = 0$ case. Thus, as presented in [4], this is thus a new two-parameter AlAdS$_5$ deformation of the AAdS$_5$ solutions of [76]. Choosing $c_2 \neq 0, c_3 = 0$ and $\mathcal{X}$ in (4.34) as $\mathcal{X} = -\frac{2\Psi}{g\tilde{m}} + \frac{c_2}{48g}I_1 I_4$, the boundary metric appears to be regular and of type Petrov III like that of [32,78].[6] Its explicit expression in the coordinates of [76] is (below we drop the label "CCLP" on the coordinates θ, ϕ, and ψ):

$$\mathrm{d}s^2_{\text{bdry}} = \mathrm{d}s^2_{\text{bdry,CCLP}} + \mathrm{d}s^2_{c_2}\ , \tag{4.45}$$

[5] Note that the present orthotoric form of the solution in [76], which is adapted to supersymmetry, does not use the same coordinates of the Plebański-Demiański-like form appearing in [115].

[6] See [116] for a discussion of the Petrov type of supersymmetric boundaries.

where the undeformed boundary metric of [76], obtained sending $gr \to \infty$, is

$$\mathrm{d}s^2_{\mathrm{bdry,CCLP}} \;=\; -\frac{\Delta_\theta}{\Xi_a \Xi_b}\mathrm{d}t^2_{\mathrm{CCLP}} + \frac{1}{g^2}\left(\frac{\mathrm{d}\theta^2}{\Delta_\theta} + \frac{\sin^2\theta}{\Xi_a}\mathrm{d}\phi^2 + \frac{\cos^2\theta}{\Xi_b}\mathrm{d}\psi^2\right), \tag{4.46}$$

with $\Xi_a = 1 - a^2g^2$, $\Xi_b = 1 - b^2g^2$ and

$$\Delta_\theta = 1 - a^2g^2\cos^2\theta - b^2g^2\sin^2\theta\ , \tag{4.47}$$

while the deformation is linear in c_2 and reads

$$\begin{aligned}
\mathrm{d}s^2_{c_2} = c_2 &\frac{g^2\tilde{m}^2\left(a^2-b^2\right)^2}{1536\Xi_a^3\Xi_b^3}\big(gt_{\mathrm{CCLP}}(\Xi_a+\Xi_b) - \Xi_b\phi - \Xi_a\psi\big)\\
&\times\big(-g\,\mathrm{d}t_{\mathrm{CCLP}}\,(\Xi_a-\Xi_b) - \Xi_b\mathrm{d}\phi + \Xi_a\mathrm{d}\psi\big)\\
&\times\big(-(\Xi_a\cos^2\theta + \Xi_b\sin^2\theta)g\,\mathrm{d}t_{\mathrm{CCLP}} + \Xi_a\cos^2\theta\mathrm{d}\psi + \Xi_b\sin^2\theta\mathrm{d}\phi\big)\ .
\end{aligned} \tag{4.48}$$

It would be interesting to study further the regularity properties of these deformations and see if they generalize the similar solutions of [32, 78, 79].

Note that both the change of coordinates (4.41) and the polynomials (4.43) are singular in the limits $\tilde{m} \to 0$ or $b \to a$, while they remain finite when $b \to -a$. (When we take $b \to \pm a$, it is understood that we keep m, and not $\tilde{m}$, fixed). We clarify the singular limits in the next section.

4.2.5 A scaling limit

In the following we show that a simple scaling limit of the orthotoric metric yields certain *non*-orthotoric Kähler metrics, that have previously been employed to construct supersymmetric solutions. We recover on the one hand the base metric considered in [79], and on the other hand an $SU(2)\times U(1)$ invariant Kähler metric. This proves that our orthotoric ansatz captures all known supersymmetric solutions to minimal five-dimensional gauged supergravity belonging to the timelike class. The procedure will also clarify the singular limits pointed out at the end of the previous subsection.

We start by redefining three of the four orthotoric coordinates $\{\eta, \xi, \Phi, \Psi\}$ as

$$\Phi = \varepsilon\,\phi\ , \qquad \Psi = \varepsilon\,\psi\ , \qquad \xi = -\varepsilon^{-1}\rho\ , \tag{4.49}$$

where ε is a parameter that we will send to zero. For the metric to be well-behaved in the limit, we also assume that the functions $\mathcal{F}$, $\mathcal{G}$ satisfy

$$\mathcal{G}(\eta) = \varepsilon^{-1}\widetilde{\mathcal{G}}(\eta) + \mathcal{O}(1)\ , \qquad \mathcal{F}(\xi) = \varepsilon^{-3}\widetilde{\mathcal{F}}(\rho) + \mathcal{O}(\varepsilon^{-2})\ , \tag{4.50}$$

where $\widetilde{\mathcal{G}}(\eta)$, $\widetilde{\mathcal{F}}(\rho)$ are independent of ε and thus remain finite in the limit. Plugging these in the orthotoric metric (4.19) and sending $\varepsilon \to 0$ we obtain

$$g^2 \mathrm{d}s_B^2 \;=\; g^2 \lim_{\varepsilon\to 0} \mathrm{d}s^2_{\mathrm{ortho}} \;=\; \frac{\rho}{\widetilde{\mathcal{F}}(\rho)}\mathrm{d}\rho^2 + \frac{\widetilde{\mathcal{F}}(\rho)}{\rho}(\mathrm{d}\phi + \eta\,\mathrm{d}\psi)^2 + \rho\left(\frac{\mathrm{d}\eta^2}{\widetilde{\mathcal{G}}(\eta)} + \widetilde{\mathcal{G}}(\eta)\mathrm{d}\psi^2\right) . \tag{4.51}$$

This is a Kähler metric of Calabi type (see *e.g.* [111]), with associated Kähler form

$$X^1 \;=\; -\frac{1}{g^2}\,\mathrm{d}\left[\rho(\mathrm{d}\phi + \eta \mathrm{d}\psi)\right] . \tag{4.52}$$

At this stage the functions $\widetilde{\mathcal{F}}(\rho)$ and $\widetilde{\mathcal{G}}(\eta)$ are arbitrary. Of course, for (4.51) to be the base of a supersymmetric solution we still need to impose on $\widetilde{\mathcal{F}}(\rho)$, $\widetilde{\mathcal{G}}(\eta)$ the equation following from the constraint (4.18).

We next consider two subcases: in the former we fix $\widetilde{\mathcal{F}}$ and recover the metric studied in [79], while in the latter we fix $\widetilde{\mathcal{G}}$ and obtain an $SU(2)\times U(1)$ invariant metric.

Case 1. We take $\widetilde{\mathcal{F}}(\rho) = 4\rho^3 + \rho^2$ and subsequently redefine $\rho = \frac{1}{4}\sinh^2(g\sigma)$. Then (4.51) becomes

$$\mathrm{d}s_B^2 \;=\; \mathrm{d}\sigma^2 + \frac{1}{4g^2}\sinh^2(g\sigma)\left(\frac{\mathrm{d}\eta^2}{\widetilde{\mathcal{G}}(\eta)} + \widetilde{\mathcal{G}}(\eta)\mathrm{d}\psi^2 + \cosh^2(g\sigma)(\mathrm{d}\phi + \eta\mathrm{d}\psi)^2\right) , \tag{4.53}$$

which is precisely the metric appearing in eq. (7.8) of [79] (upon identifying $\eta = x$ and $\widetilde{\mathcal{G}}(\eta) = H(x)$). In this case our equation (4.18) becomes

$$\left(\widetilde{\mathcal{G}}^2\, \widetilde{\mathcal{G}}''''\right)'' \;=\; 0 , \tag{4.54}$$

that coincides with the constraint found in [79]. As discussed in [79], this Kähler base metric supports the most general supersymmetric black hole solution free of CTC's that is known within minimal five-dimensional gauged supergravity. This is obtained from the supersymmetric solutions of [76] by setting $\tilde{m} = 0$. In fact, the limit $\tilde{m} \to 0$ in the map (4.41), (4.43) is an example of the present $\varepsilon \to 0$ limit, where the resulting $\widetilde{\mathcal{G}}(\eta)$ is a cubic polynomial [77, 79].[7] Particular non-polynomial solutions to eq. (4.54) were found in [79], however in the same paper these were shown to yield unacceptable singularities in the five-dimensional metric.

[7] This can be seen starting from (4.41), (4.43) and redefining $\tilde{m} = -\frac{8\alpha^2}{(a^2-b^2)}\varepsilon$ and $r^2 = r_0^2 + 4\alpha^2\rho$, where we are denoting $\alpha^2 = r_0^2 + \frac{(1+ag+bg)^2}{g^2}$ and $r_0^2 = \frac{a+b+abg}{g}$. It follows that $\xi = \varepsilon^{-1}\rho + \mathcal{O}(1)$. Then implementing the scaling limit described above we get $\widetilde{\mathcal{F}}(\rho) = 4\rho^3 + \rho^2$ and $\widetilde{\mathcal{G}}(\eta) = \frac{1}{2}(1-\eta^2)\left[A_1^2 + A_2^2 + (A_1^2 - A_2^2)\eta\right]$ with $A_1^2 = \frac{1-a^2g^2}{g^2\alpha^2}$ and $A_2^2 = \frac{1-b^2g^2}{g^2\alpha^2}$. This makes contact with the description of the supersymmetric black holes of [76] given in [77, 79].

Case 2. If instead we take $\widetilde{\mathcal{G}}(\eta) = 1 - \eta^2$ and redefine $\eta = \cos\theta$, then the metric (4.51) becomes

$$g^2\, \mathrm{d}s^2_B \;=\; \frac{\rho}{\widetilde{\mathcal{F}}(\rho)}\mathrm{d}\rho^2 + \frac{\widetilde{\mathcal{F}}(\rho)}{\rho}(\mathrm{d}\phi + \cos\theta\, \mathrm{d}\psi)^2 + \rho\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\psi^2\right) \;, \tag{4.55}$$

with Kähler form

$$X^1 \;=\; -\frac{1}{g^2}\,\mathrm{d}\left[\rho(\mathrm{d}\phi + \cos\theta \mathrm{d}\psi)\right] \;. \tag{4.56}$$

This has enhanced $SU(2)\times U(1)$ symmetry compared to the $U(1)\times U(1)$ invariant orthotoric metric. It is in fact the most general Kähler metric with such symmetry and is equivalent, by a simple change of variable, to the metric ansatz employed in [72] to construct the first supersymmetric AAdS black hole free of CTC's. The constraint (4.18) becomes a sixth-order equation for $\widetilde{\mathcal{F}}(\rho)$. This is explicitly solved if $\widetilde{\mathcal{F}}(\rho)$ satisfies the fifth-order equation

$$\begin{aligned} &16(\widetilde{\mathcal{F}}')^2 + 4\rho^2\left(6\widetilde{\mathcal{F}}'' + (\widetilde{\mathcal{F}}'')^2 - 2\rho\widetilde{\mathcal{F}}^{(3)}\right) + 2\rho\widetilde{\mathcal{F}}'\left(-24 - 4\widetilde{\mathcal{F}}'' - 4\rho\widetilde{\mathcal{F}}^{(3)} + 3\rho^2\widetilde{\mathcal{F}}^{(4)}\right) \\ &-3\widetilde{\mathcal{F}}\left(-16 + 8\widetilde{\mathcal{F}}'' - 8\rho\widetilde{\mathcal{F}}^{(3)} + 4\rho^2\widetilde{\mathcal{F}}^{(4)} - \rho^3\widetilde{\mathcal{F}}^{(5)}\right) \;=\; 0 \;. \end{aligned} \tag{4.57}$$

Upon a change of variable, the latter is equivalent to the sixth-order equation presented in [72, eq. (4.23)]. It was proved there that a solution completely specifies an $SU(2)\times U(1)$ invariant five-dimensional metric and graviphoton. We find that a simple solution to (4.57) is provided by a cubic polynomial

$$\widetilde{\mathcal{F}}(\rho) = f_0 + f_1\rho + f_2\rho^2 + f_3\rho^3 \;, \quad \text{such that} \quad f_1^2 + 3f_0(1-f_2) = 0 \;. \tag{4.58}$$

Supersymmetric AAdS solutions with $SU(2)\times U(1)$ symmetry were also found in [75] and further discussed in [109]. It is easy to check that after scaling away a trivial parameter, the five-dimensional solution determined by (4.58) in fact reproduces[8] the two-parameter "case B" solution given in [109, sect. 3.4]. In turn, the latter includes the black hole of [72], and a family of topological solitons for particular values of the parameters.

The special case $f_1 = 0$, $f_2 = 1$ yields the most general Kähler-Einstein metric with $SU(2)\times U(1)$ isometry; this has curvature $R = -6g^2 f_3$ and is diffeomorphic to the Bergmann metric only for $f_0 = 0$. The corresponding $SU(2)\times U(1)$ invariant five-dimensional solution is "Lorentzian Sasaki-Einstein": for $f_0 = 0$ this is just AdS_5, while for $f_0 \neq 0$ it features a curvature singularity at $\rho = 0$.

In [68], a different solution of equation (4.57) was put forward, leading to a smooth AlAdS five-dimensional metric. The non-conformally flat boundary is given by a squashed $S^3\times\mathbb{R}$, where the squashing is along the Hopf fibre and thus preserves

[8]In the case the charges are set equal, so that the two vector multiplets of the $U(1)^3$ gauged theory can be truncated away and the solution exists within minimal gauged supergravity.

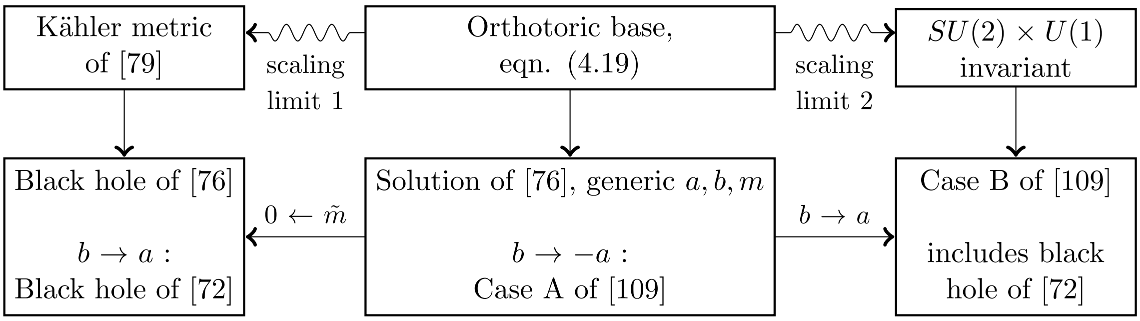

Figure 4.1: Kähler base metrics (above) and corresponding known AAdS solutions (below), with relevant references.

$SU(2) \times U(1)$ symmetry.

A particular example of this $\varepsilon \to 0$ limit is given by the $b \to a$ limit in the map (4.41), (4.43) relating the solution of [76] and the one based on our orthotoric ansatz.[9] In fact, taking $b = a$ in the solutions of [76] yields precisely the solutions presented in [109, sect. 3.4].

Note that since the black hole of [72] is obtained from the general solution of [76] by taking $\tilde{m} = 0$ and $b = a$, it belongs both to our cases 1 and 2.

In figure 4.1 we summarise the relation between different Kähler metrics and the corresponding AAdS solutions in five dimensions.

4.3 Topological solitons

In this section we focus on a sub-family of the solution of [76], known as "topological solitons" with non-trivial geometry but no horizon. *A priori*, these may be considered as candidate gravity dual to pure states of SCFTs defined on $\mathbb{R} \times S^3$. In section 4.3.1 we consider the non-vanishing vacuum expectation values of the energy and R-charge of such theories, and we look for a possible gravity dual. The constraints from the superalgebra naturally lead us to consider a 1/2 BPS topological soliton, however a direct comparison of the charges with the SCFT vacuum expectation values shows that these do not match. In section 4.3.2 we argue that in the dual SCFT certain background R-symmetry field must be turned on, implying a constraint on the R-charges and suggesting that the state dual to the topological soliton is different from the vacuum.

[9] This can be seen starting from (4.41), (4.43), redefining $b = a + 8(1 - a^2 g^2)\big[\frac{g^3 m}{(1+2ag)(1+ag)^2} - 2ag^2\big]^{-1}\varepsilon$ after having re-expressed $\tilde{m}$ as in (4.42), and implementing the scaling limit. This gives $\widetilde{\mathcal{G}}(\eta) = 1 - \eta^2$ and a cubic polynomial $\widetilde{\mathcal{F}}(\rho)$ satisfying (4.58).

4.3.1 Comparison with the supersymmetric Casimir energy

In this section we assess the possible relevance of the supergravity solutions discussed above to account for the vacuum state of dual four-dimensional $\mathcal{N} = 1$ SCFTs defined on the cylinder $\mathbb{R} \times S^3$, discussed in chapter 3. We consider the conformally flat background with the round S^3. Moreover, the background includes a non-dynamical flat gauge field A^{cs} coupling to the R-current. This is chosen such that half of the eight supercharges in the superconformal algebra commute with the Hamiltonian generating time translations on the cylinder. This ensures that the four charges are preserved when Euclidean time is compactified to a circle.

Recall from section 3.5 that the Hamiltonian H_{susy} is related to the operator Δ generating dilatations in flat space as $H_{\text{susy}} = \Delta + \frac{1}{2r_3}R$, where R is the R-charge operator. We found in that section the vacuum expectation values of the bosonic charges,

$$\begin{aligned} \langle H_{\text{susy}} \rangle &\equiv \langle \Delta \rangle + \frac{1}{2r_3}\langle R \rangle = -\frac{1}{r_3}\langle R \rangle = \frac{4}{27r_3}(\mathbf{a} + 3\mathbf{c}) , \\ \langle J_3 \rangle &= 0 , \end{aligned} \tag{4.59}$$

where $\mathbf{a}, \mathbf{c}$ are the trace anomaly coefficients, and J_3 is the conserved charge of the left $U(1)_l \subset SU(2)_l \times SU(2)_r$ isometry group. As was discussed in chapter 3, the *a priori* divergent anomaly coefficients $\mathbf{a}, \mathbf{c}$ are free of ambiguities as long as their regularization does not break supersymmetry.

Based on the above information we infer that the five-dimensional gravity dual should be AAdS$_5$ and preserve (at least) four supercharges. It should allow for a graviphoton A behaving as $A \to c\mathrm{d}t$ at the boundary, where c is a constant chosen such that the asymptotic Killing spinors generating the supersymmetry algebra (3.125) are independent of time. Indeed, the general Killing spinor of AdS$_5$ that solves the Killing spinor equation (4.3) reduces to a Weyl spinor on the boundary of the form,

$$\epsilon \xrightarrow[r\to\infty]{} \epsilon = (gr)^{1/2}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}(\sqrt{3}c+1)gt}\zeta + \mathrm{e}^{\frac{\mathrm{i}}{2}(\sqrt{3}c-1)gt}\chi\right) , \tag{4.60}$$

where ζ and χ are spinors independent of t and the radial coordinate r. We see that choosing $c = \pm\frac{1}{\sqrt{3}}$, half the spinors are independent of time. Note that if we Wick rotate $t \to -\mathrm{i}\tau$ and compactify the time coordinate, the other half is not well-defined. Hence we should regard Euclidean AAdS$_5$ spaces (including pure AdS$_5$) with compact $S^1 \times S^3$ boundary as preserving at most four supercharges.

In the context of type IIB supergravity on Sasaki-Einstein five-manifolds, we can translate the value of the vacuum energy and R-charge given in (4.59) into gravity units using the standard dictionary $\mathbf{a} = \mathbf{c} = \frac{\pi^2}{g^3\kappa_5}$. We shall also fix the radius of the boundary S^3 to $r_3 = 1/g$ for simplicity. Finally, we map the field theory vevs into supergravity charges as $\langle\Delta\rangle = E$, $\langle R\rangle = \frac{1}{\sqrt{3}g}Q$ and $\langle J_3\rangle = J_{\text{left}}$, where E is the total gravitational energy, Q the electric charge under the graviphoton and J_{left} the left angular momentum. We thus obtain the following expected values for the charges of the dual gravity solution:[10]

$$E \;=\; -\frac{\sqrt{3}}{2}Q \;=\; \frac{8}{9}\frac{\pi^2}{g^2\kappa_5^2}\;, \qquad\qquad J_{\text{left}} \;=\; 0\;. \tag{4.61}$$

The relation between E and Q and the vanishing J_{left} are indeed consistent with the anticommutation relation for the preserved AdS supercharges [109]

$$\{\mathcal{Q}_{\text{sugra}}, \mathcal{Q}^\dagger_{\text{sugra}}\} \;=\; E + \frac{\sqrt{3}}{2}Q + 2g\sigma^i J^i_{\text{left}}\;. \tag{4.62}$$

The generators J_{right} of $SU(2)_r$ appears instead in the anticommuntator of the broken supercharges. While there exists different prescriptions for the computation of the energy in asymptotically AdS spacetimes, here we will require E to be related to Q as dictated by the superalgebra. We evaluate the charge Q by the standard formula

$$Q \;=\; \frac{1}{\kappa_5^2}\int_{S^3} *_5 F\;, \tag{4.63}$$

where the integral is over the three-sphere at the boundary. In general, (4.63) contains an additional $A\wedge F$ term, but in the present case such a term does not contribute, since we take $F\to 0$ asymptotically.

The obvious candidate to describe the vacuum of the dual SCFT is global AdS$_5$. Indeed, the boundary is $\mathbb{R}\times S^3$, and one may turn on a constant component for the graviphoton $A_t = c$. However, since $F = 0$ everywhere, clearly $Q = 0$ from (4.63). One possible solution to this mismatch with (4.61) may come from a careful analysis of how the evaluation of charges is compatible with supersymmetry. Here we will not address this question further. Instead we will consider a supersymmetric solution among those discussed in this chapter with a graviphoton so that (4.63) yields a non-vanishes charge. Although we do not find agreement for the holographic charges below, we do clarify certain aspects of such solutions.

As we consider the asymptotically flat boundary $\mathbb{R}\times S^3$, this sets $c_2 = c_3 = 0$. In this case the solutions in this chapter reduce to the supersymmetric solutions of [76], controlled by the three parameters a, b, m. To match the SCFT on the boundary, the solution should preserve four supercharges, that is, it should be 1/2-BPS. For the solution in [76], this was shown to be the case when $a + b = 0$.[11] This identifies a two-parameter family of solutions with $SU(2)\times U(1)$ invariance, originally found

[10] Recall that E is *not* the same as $E_{\text{susy}} = \langle H_{\text{susy}}\rangle$, but the two quantities are related as $E = \langle\Delta\rangle = \frac{3}{2}\langle H_{\text{susy}}\rangle = \frac{3}{2}E_{\text{susy}}$.

[11] In fact, it was shown in [4] that while the solution of [76] also contains a 1/4-BPS topological soliton, this is plagued by conical singularities. Only the 1/2-BPS topological soliton with $a+b=0$ is completely regular.

in [74] and further studied in [109][12].

As mentioned previously, the change of coordinates (4.41) remains finite in the limit $b \to -a$ at fixed m. However, we will use the coordinates $\{\mathfrak{t}, \theta, \phi, \psi, r\}$ of [109], related to the orthotoric coordinates as

$$t = \mathfrak{t} \ , \quad \eta = \cos\theta \ , \quad \xi = \frac{r^2}{\alpha g} \ , \quad \Phi = \frac{\alpha g^3}{4}(\phi + 2g\mathfrak{t}) \ , \quad \Psi = \frac{\alpha g^3}{4}\psi \ , \tag{4.64}$$

where we renamed the parameters,

$$a \ = \ \frac{\alpha}{q} \ , \qquad m \ = \ \frac{(q^2 - \alpha^2 g^2)^2}{q^3} \ . \tag{4.65}$$

In these coordinates, the five-dimensional metric reads

$$\mathrm{d}s_5^2 \ = \ -\frac{r^2\mathcal{U}}{4\mathcal{B}}\mathrm{d}\mathfrak{t}^2 + \frac{\mathrm{d}r^2}{\mathcal{U}} + \mathcal{B}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi + \mathfrak{f}\,\mathrm{d}\mathfrak{t})^2 + \frac{1}{4}(r^2+q)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \ , \tag{4.66}$$

where

$$\mathcal{U} \ = \ \frac{r^4 + g^2(r^2+q)^3 - g^2\alpha^2}{r^2(r^2+q)} \ , \qquad \mathcal{B} \ = \ \frac{(r^2+q)^3 - \alpha^2}{4(r^2+q)^2} \ , \qquad \mathfrak{f} \ = \ \frac{2\alpha r^2}{\alpha^2 - (r^2+q)^3} \ , \tag{4.67}$$

and the graviphoton is

$$A \ = \ \frac{\sqrt{3}}{r^2+q}\left(q\,\mathrm{d}\mathfrak{t} - \frac{1}{2}\alpha\,(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi)\right) + c\,\mathrm{d}\mathfrak{t} \ . \tag{4.68}$$

Here, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, $\psi \in [0, 4\pi)$ are the standard Euler angles parametrizing the three-sphere of $\mathbb{R} \times S^3$ at the boundary at $r \to \infty$.

We can now discuss the charges. From (4.63), the charge under the graviphoton is found to be

$$Q \ = \ -4\sqrt{3}q\frac{\pi^2}{\kappa_5^2} \ . \tag{4.69}$$

The angular momentum conjugate to a rotational Killing vector k^μ is given by the Komar integral $J = \frac{1}{2\kappa_5^2}\int_{S^3} *_5\mathrm{d}k$, where $k = k_\mu \mathrm{d}x^\mu$. For the angular momentum J_{left} conjugate to $\partial/\partial\phi$, we get

$$J_{\mathrm{left}} = 0 \ , \tag{4.70}$$

while J_{right}, conjugate to $\partial/\partial\psi$, is controlled by α,

$$J_{\mathrm{right}} \ = \ 2\alpha\frac{\pi^2}{\kappa_5^2} \ . \tag{4.71}$$

The energy was computed in [109] by integrating the first law of thermodynamics,

[12] See the "case A" in section 3.3 of [109], wih all the charges set equal $q_1 = q_2 = q_3 = q$, so that the solution fits with minimal gauged supergravity.

yielding

$$E = -\frac{\sqrt{3}}{2}Q = 6q\frac{\pi^2}{\kappa_5^2} , \tag{4.72}$$

which is in agreement with the superalgebra (4.62). Whether (4.72) matches the expectation from (4.61) depends on the parameter q. To see how q should be fixed, we need to discuss the global structure of the solution.

Let us first observe that by setting the rotational parameter $\alpha = 0$, the $SU(2) \times U(1)$ symmetry of (4.66), (4.68) is enhanced to $SO(4)$. This solution was originally found in [73] and contains a naked singularity for any value of $q \neq 0$. So while the $\alpha = 0$ limit provides the natural symmetries to describe the vacuum of an SCFT on $\mathbb{R} \times S^3$, it yields a solution that for any $q \neq 0$ is pathological, at least in supergravity. In appendix D we prove that there are no other supersymmetric solutions with $\mathbb{R} \times SO(4)$ symmetry within minimal gauged supergravity.

It was shown in [109] that the two-parameter family of solutions given by (4.66), (4.68) contains a regular topological soliton (while there are no black holes free of CTC's). This is obtained by tuning the rotational parameter α to the critical value

$$\alpha^2 = q^3 . \tag{4.73}$$

Then the metric (4.66) has no horizon, is free of CTC's, and extends from $r = 0$ to infinity. In addition, for the r, ψ part of the metric to avoid a conical singularity as $r \to 0$, one has to impose

$$q = \frac{1}{9g^2} . \tag{4.74}$$

In this way one obtains a spin^c manifold with topology $\mathbb{R} \times (\mathcal{O}(-1) \to S^2)$, where the first factor is the time direction, and the second has the topology of Taub-Bolt space [109]. Since $\frac{\sqrt{3}}{2}gA$ is a connection on a spin^c bundle, as it can be seen from (4.3), one must also check the quantization condition for the flux threading the two-cycle at $r = 0$. This reads

$$\frac{1}{2\pi}\frac{\sqrt{3}}{2}g\int_{S^2} F \in \mathbb{Z} + \frac{1}{2}, \tag{4.75}$$

where the quantization in half-integer units arises because the manifold is spin^c rather than spin. One can check that

$$\frac{1}{2\pi}\frac{\sqrt{3}}{2}g\int_{S^2} F = \frac{3}{2}g\, q^{1/2} = \frac{1}{2} , \tag{4.76}$$

hence the condition is satisfied.

We can then proceed to insert (4.74) into (4.69). This gives

$$E = -\frac{\sqrt{3}}{2}Q = \frac{2}{3}\frac{\pi^2}{g^2\kappa_5^2} , \tag{4.77}$$

which is different from (4.61). In field theory units, this gives $\langle R \rangle = -\frac{4}{9}\mathbf{a} \neq -\frac{16}{27}\mathbf{a}$, where the latter is the vev of the R-charge in a supersymmetric vacuum [3] (recall footnote 10). We conclude that although this 1/2 BPS topological soliton is smooth and seemingly fulfills the requirements imposed by the field theory superalgebra, it is not dual to the vacuum state of an SCFT on the $\mathbb{R} \times S^3$ background. Below we will give further evidence that this solution cannot describe the supersymmetric vacuum state of a generic SCFT on $\mathbb{R} \times S^3$.

4.3.2 Further remarks on supersymmetric topological solitons

We found above that a direct comparison of the holographic charges of the 1/2-BPS topological soliton with the charges of SCFTs on $\mathbb{R} \times S^3$ did not provide a match. We now briefly discuss further arguments that this gravity solution cannot be the dual of such field theories.

Firstly, we note that the non-trivial topology of the solution presents an obstruction to its embedding into string theory, precisely analogous to the situation of the "bolt solutions" found in [63]. Locally, all solutions to five-dimensional minimal gauged supergravity can be embedded into type IIB supergravity on a Sasaki-Einstein five-manifold [117], however, one may encounter global obstructions when the topology of the external space has non-trivial topology. It was noted in [109] that the topological soliton cannot be uplifted on S^5. An uplift on the Lens space $S^5/\mathbb{Z}_p$ was discussed in the appendix of [4], including examples. We refer to this reference for the details.

In the present context, the global uplift provides further evidence that the 1/2-BPS topological soliton is not the gravity dual of SCFTs on $\mathbb{R} \times S^3$. In the gauge/gravity correspondence, different SCFTs on $\mathbb{R} \times S^3$ are dual to gravity solutions uplifted on different internal manifolds. While the topological soliton can be uplifted only on specific internal manifolds, the vacuum state of the SCFTs considered in [3] leading to (4.59) exists for any such SCFTs. Therefore the gravity solution cannot be the correct dual description.

Furthermore, it was shown in [4] that the R-charges q_R of fields in an SCFT on the boundary of the 1/2-BPS topological soliton must satisfy a quantization condition $q_R \in 2\mathbb{Z}$. This condition follows since it is necessary to cover the topological soliton by two gauge patches. Again such a constraint on the R-charges is not present for the SCFTs on $\mathbb{R} \times S^3$, leading to the conclusion that the topological soliton is not the correct gravity dual.

4.4 Conclusions

In this chapter we studied supersymmetric solutions to minimal gauged supergravity in five dimensions via the approach of [32]. We derived the general expression (4.18) for the sixth-order constraint that must be satisfied by the Kähler base metric in the timelike class. We then considered a general ansatz comprising an orthotoric Kähler base (4.19), for which the constraint reduced to a single sixth-order equation for two functions, each of one variable. We succeeded in finding an analytic solution to this equation, yielding a family of AlAdS solutions with five non-trivial parameters. We showed that after setting two of the parameters to zero, such that the solution is AAdS, the solution reduces to that of [76]. Hence, this ansatz encompasses all known supersymmetric AAdS_5 solutions of minimal gauged supergravtiy in the timelike class (taking into account the scaling limits mentioned at the end of section 4.2.4). This highlights the role of orthotoric Kähler metrics in providing supersymmetric solutions to five-dimensional gauged supergravity. For general values of the five non-trivial parameters, we obtained an AlAdS generalization of the solutions of [76], of the type previously presented in [32, 78, 79] in more restricted setups. There exists a further generalization by an arbitrary anti-holomorphic function [32]; it would be interesting to study regularity and global properties of these AlAdS solutions.

It would also be interesting to investigate further the existence of solutions to our "master equation" (4.31), perhaps aided by numerical analysis. In particular, our orthotoric setup could be used as the starting point for constructing a supersymmetric AlAdS solution dual to SCFT's on a squashed $\mathbb{R} \times S^3$ background, where the squashing of the three-sphere preserves just $U(1) \times U(1)$ symmetry. This would generalize the $SU(2) \times U(1)$ invariant solution of [68].

Finally, we have discussed the possible relevance of the solutions above to account for the non-vanishing supersymmetric vacuum energy and R-charge of a four-dimensional $\mathcal{N} = 1$ SCFT defined on the cylinder $\mathbb{R} \times S^3$. The most obvious candidate for the gravity dual to the vacuum of an SCFT on $\mathbb{R} \times S^3$ is AdS_5 in global coordinates; however this comes with a vanishing R-charge. In appendix D we have performed a complete analysis of supersymmetric solutions with $\mathbb{R} \times SO(4)$ symmetry, proving that there exists a unique singular solution, where the charge is an arbitrary parameter [73]. We then focused on the 1/2 BPS smooth topological soliton of [109], however, a direct evaluation of the energy and electric charge showed that these do not match the SCFT vacuum expectation values.

We cannot exclude that there exist other solutions, possibly within our orthotoric ansatz, or perhaps in the null class of [32], that match the supersymmetric Casimir energy of a four-dimensional $\mathcal{N} = 1$ SCFT defined on the cylinder $\mathbb{R} \times S^3$. It would also be worth revisiting the evaluation of the charges of empty AdS space, and see if suitable boundary terms can shift the values of both the energy and electric charge, in a way compatible with supersymmetry.

Chapter 5

Conclusions

This thesis includes work on both sides of the gauge/gravity duality. We constructed in chapter 2 the gravity duals of supersymmetric field theories defined on a broad class of three-manifolds. These gravity duals are supersymmetric solutions of four-dimensional minimal gauged supergravity, comprising a self-dual Einstein metric on the four-ball and the anti-self-dual graviphoton. We computed the holographically renormalized on-shell action (2.92), finding that it depends on the background only through one parameter, b_1/b_2, describing the supersymmetric Killing vector. The concrete check was the match with the field theory free energy (1.25), obtained previously using localization.

This work widens the class of known examples of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality, and it would be interesting to study in more detail the explicit m-pole solutions described in section 2.4.4. A number of generalizations are discussed in section 2.5. One may relax the conditions that the graviphoton is both real and anti-self-dual. Indeed, while the boundary is smooth for any choice of b_1/b_2, we found that the gravity solution is regular only if $b_1/b_2 > 0$ or $b_1/b_2 = -1$. It is natural to expect that for the remaining choices of b_1/b_2 the boundaries can be filled by gravity solutions with non-self-dual graviphoton. Beyond this, one may consider geometries of more general topology. One related development is the holographic computation of the entropy of a class of supersymmetric asymptotically AdS_4 black holes [118]. These black holes are solutions of four-dimensional $\mathcal{N} = 2$ gauged supergravity coupled to vector multiplets, and were first found analytically in [119]. In [118], the entropy is computed from a topologically twisted index for ABJM theory on $S^1 \times S^2$ in the large N limit, providing for the first time a microscopic interpretation for the entropy of an AAdS black hole. It would interesting to extend this to other supersymmetric black holes in four and other dimensions.

On the field theory side, we studied in chapter 3 the Casimir energy of $\mathcal{N} = 1$ field theories. By canonically quantizing the Hamiltonian, we clarified that the Casimir energy of free CFTs and the supersymmetric Casimir energy E_{susy} arise as the expectation values of two *different* Hamiltonians using the *same* zeta function regularization. By reducing the theory on S^3 to a one-dimensional theory, we then showed that in fact E_{susy} is unambiguously defined, provided the regularization scheme preserves supersymmetry.

There has recently been further work on the supersymmetric Casimir energy. Ref. [120] studied SCFTs on backgrounds of topology $S^1 \times S^{d-1}$ with $d = 2, 4, 6$, where it was conjectured that the supersymmetric Casimir energy can be computed using the equivariant anomaly polynomial. For the four-dimensional $\mathcal{N} = 1$ field theories discussed in section 1.2.1, the connection of E_{susy} to anomaly polynomials on $S^1 \times S^3$ was recently explained in [121]. This paper studied such theories on more general backgrounds $S^1 \times M_3$, with M_3 a compact three-manifold, and it was found that the supersymmetric Casimir energy is computed as a limit of the index-character counting holomorphic functions. Besides these developments, it would be interesting to apply the approach of chapter 3 to the six-dimensional case $\mathbb{R} \times S^5$, and squashed versions thereof. It should also be possible to consider more general topologies $\mathbb{R} \times M_5$. For example, the localized partition function for five-dimensional super-Yang-Mills defined on toric Sasaki-Einstein manifolds Y_5 has been computed in [122]. From this result it should be possible to obtain the partition function for six-dimensional theories on $S^1 \times Y_5$ and subsequently study the supersymmetric Casimir energy.

As a physical quantity, the supersymmetric Casimir energy should have a holographic interpretation. With this in mind, we constructed in chapter 4 new supersymmetric AlAdS$_5$ solutions of five-dimensional minimal gauged supergravity from an ansatz based on an orthotoric Kähler metric. It would be interesting to study further the properties of these solutions. Our solution also recovered (including the scaling limits) all known supersymmetric AAdS$_5$ solutions of this theory. However, our investigation of whether the 1/2-BPS topological soliton could be the gravity dual of $\mathcal{N} = 1$ field theories on the conformally flat $\mathbb{R} \times S^3$ led to the conclusion that this is not the case. It thus remains an open problem to account holographically for the supersymmetric Casimir energy. There could exist more general supersymmetric solutions in the timelike class of five-dimensional minimal gauged supergravity. In this case, such solutions must solve the general constraint (4.18), first presented in [4]. Alternatively, one may have to consider solutions in the null class, or perhaps even a revision is needed of the way holographic charges are computed.

Appendix A

Spherical harmonics on S^3

A.1 Scalar spherical harmonics

In this appendix we give some details on the scalar and spinor spherical harmonics on the three-sphere, following [99,123]. We can obtain the metric on the unit three-sphere by considering a parametrization on $\mathbb{R}^4 \simeq \mathbb{C}^2$ with the metric,

$$\mathrm{d}s^2_{\mathbb{C}^2} \;=\; \mathrm{d}u\mathrm{d}\bar{u} + \mathrm{d}v\mathrm{d}\bar{v}\ . \tag{A.1}$$

The three-sphere of unit radius is then defined by

$$u\bar{u} + v\bar{v} \;=\; 1\ . \tag{A.2}$$

The isometry group is $SO(4) \simeq SU(2)_l \times SU(2)_r$, with generators[1] L^l_a and L^r_a, with $a = 1,2,3$, satisfying

$$\left[L^l_a, L^l_b\right] \;=\; \mathrm{i}\epsilon_{abc}L^l_c\ , \qquad \left[L^r_a, L^r_b\right] \;=\; \mathrm{i}\epsilon_{abc}L^r_c\ , \qquad \left[L^l_a, L^r_b\right] \;=\; 0\ . \tag{A.3}$$

As usual, we define raising and lowering operators,

$$L^l_\pm \;=\; L^l_1 \pm \mathrm{i}L^l_2\ , \qquad L^r_\pm \;=\; L^r_1 \pm \mathrm{i}L^r_2\ . \tag{A.4}$$

In the (u,v)-coordinates, these are represented by

$$\begin{aligned} L^l_+ &= -u\partial_{\bar{v}} + v\partial_{\bar{u}}\ , \quad & L^l_- &= \bar{u}\partial_v - \bar{v}\partial_u\ , \\ L^r_+ &= -u\partial_v + \bar{v}\partial_{\bar{u}}\ , \quad & L^r_- &= \bar{u}\partial_{\bar{v}} - v\partial_u\ , \end{aligned} \tag{A.5}$$

while

$$L^l_3 \;=\; \frac{1}{2}\left(u\partial_u + v\partial_v - \bar{u}\partial_{\bar{u}} - \bar{v}\partial_{\bar{v}}\right)\ , \qquad L^r_3 \;=\; \frac{1}{2}\left(u\partial_u - v\partial_v - \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}\right) \tag{A.6}$$

[1] In the main text, we use the operators L^l_a, but drop the superscript l.

In terms of these operators, the scalar Laplacian is

$$-\nabla_i\nabla^i \;=\; 4L^l_a L^l_a \;=\; 4L^r_a L^r_a \;. \tag{A.7}$$

The spherical harmonics Y_ℓ^{mn} are constructed starting from the highest weight state,

$$Y_\ell^{\frac{\ell}{2}\frac{\ell}{2}} \;=\; \sqrt{\frac{\ell+1}{2\pi^2}}\, u^\ell \;, \tag{A.8}$$

which is annihilated by the raising operators L^l_+ and L^r_+. The number m (n) can be lowered by L^l_- (L^r_-), so that

$$Y_\ell^{mn} \;\propto\; \left(L^l_-\right)^{\frac{\ell}{2}-m}\left(L^r_-\right)^{\frac{\ell}{2}-n} Y_\ell^{\frac{\ell}{2}\frac{\ell}{2}} \;, \tag{A.9}$$

and take values $-\frac{\ell}{2} \leq m,n \leq \frac{\ell}{2}$. Recall the operator $\mathcal{O}_b$ of equation (3.21),

$$\mathcal{O}_b \;=\; 2\alpha_b \vec{L}^2 + 2\beta_b L^l_3 + \gamma_b \;. \tag{A.10}$$

The spherical harmonics are eigenfunctions of this operator

$$\mathcal{O}_b Y_\ell^{mn} \;=\; E_b^2 Y_\ell^{mn} \;, \qquad E_b^2 \;=\; \frac{\alpha_b}{2}\ell(\ell+2) + 2\beta_b m + \gamma_b \;, \tag{A.11}$$

and also

$$L^l_3 Y_\ell^{mn} \;=\; m Y_\ell^{mn} \;, \qquad L^r_3 Y_\ell^{mn} \;=\; n Y_\ell^{mn} \;. \tag{A.12}$$

We normalize the spherical harmonics as [123]

$$Y_\ell^{\frac{\ell}{2}-a,\frac{\ell}{2}-b} \;=\; N_{\ell ab}\sum_k \frac{(-u)^{\ell+k-a-b}\bar{u}^k v^{b-k}\bar{v}^{a-k}}{k!(\ell+k-a-b)!(a-k)!(b-k)!} \;, \tag{A.13}$$

where the sum is over all integer values of k for which the exponents are non-negative, and

$$N_{\ell ab} \;=\; \sqrt{\frac{(\ell+1)a!b!(\ell-a)!(\ell-b)!}{2\pi^2}} \;. \tag{A.14}$$

Now specifically taking $u = \mathrm{i}\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}(\varphi-\varsigma)/2}$ and $v = \cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}(\varphi+\varsigma)/2}$, one finds the metric

$$\mathrm{d}s^2_{S^3} \;=\; \frac{1}{4}\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 + (\mathrm{d}\varsigma + \cos\theta\mathrm{d}\varphi)^2\right) \;, \tag{A.15}$$

and

$$L^l_3 \;=\; \mathrm{i}\partial_\varsigma \qquad L^r_3 \;=\; -\mathrm{i}\partial_\varphi \;. \tag{A.16}$$

With the above normalization, the spherical harmonics satisfy

$$\int \sqrt{g_3}\mathrm{d}^3x Y_\ell^{mn}\left(Y_{\ell'}^{m'n'}\right)^* \;=\; \delta_{\ell,\ell'}\delta^{m,m'}\delta^{n,n'} \;, \tag{A.17}$$

and

$$(Y_\ell^{mn})^* \;=\; (-1)^{m+n} Y_\ell^{-m,-n} \;, \tag{A.18}$$

as well as the completeness relation,

$$\sum_{\ell,m,n} Y_\ell^{mn}(\theta,\varphi,\varsigma)\,(Y_\ell^{mn}(\theta',\varphi',\varsigma'))^* \;=\; \frac{1}{\sin\theta}\delta^{(3)}(\vec{x}-\vec{x}\,') \;, \tag{A.19}$$

where $\delta^{(3)}(\vec{x}-\vec{x}\,') = \delta(\theta-\theta')\delta(\varphi-\varphi')\delta(\varsigma-\varsigma')$.

A.2 Spinor spherical harmonics

The spinor spherical harmonics can be constructed from the scalar harmonics. These are eigenspinors of the operator

$$\mathcal{O}_f \;=\; 2\alpha_f \vec{L}\cdot\vec{S} + 2\beta_f S_3 + \gamma_f \;, \tag{A.20}$$

where L_a are the left-invariant operators of the previous subsection, and $S_a = \frac{\gamma_a}{2}$, where γ_a are the Pauli matrices. For $\beta_f = 0$, the spinor spherical harmonics can be constructed as [99]

$$S^{\pm}_{\ell mn} \;=\; \begin{pmatrix} \cos\nu^{\pm}_{\ell m}\, Y_\ell^{mn} \\ \sin\nu^{\pm}_{\ell m}\, Y_\ell^{m+1,n} \end{pmatrix} \;, \tag{A.21}$$

where

$$\sin\nu^{\pm}_{\ell m} \;=\; \mp\sqrt{\frac{\ell+1\pm(2m+1)}{2(\ell+1)}} \;, \qquad \cos\nu^{\pm}_{\ell m} \;=\; \sqrt{\frac{\ell+1\mp(2m+1)}{2(\ell+1)}} \;. \tag{A.22}$$

For $S^{+}_{\ell mn}$, one has $\ell \geq 1$ and $-\frac{\ell}{2} \leq m \leq \frac{\ell}{2}-1$, while for $S^{-}_{\ell mn}$ one has $\ell \geq 0$ and $-\frac{\ell}{2}-1 \leq m \leq \frac{\ell}{2}$. In both cases $-\frac{\ell}{2} \leq n \leq \frac{\ell}{2}$. The spinor spherical harmonics satisfy the completeness relation

$$\sum_{m,n} S^{\pm}_{\ell mn\,\alpha}(x)\,\left(S^{\pm}_{\ell mn}(x)\right)^{\dagger}_{\dot\alpha} \;=\; \frac{1}{4\pi^2} n^{\pm}_{\ell}\,\mathbb{1}_{\alpha\dot\alpha} \;, \tag{A.23}$$

with $n^{+}_{\ell} = \ell(\ell+1)$ and $n^{-}_{\ell} = (\ell+2)(\ell+1)$. Further, using the properties of Y_ℓ^{mn}, one can show the identities

$$\sum_{\ell,m,n}\left[S^{+}_{\ell mn\,\alpha}(x)\,\left(S^{+}_{\ell mn}(x')\right)^{\dagger}_{\dot\alpha} + S^{-}_{\ell mn\,\alpha}(x)\,\left(S^{-}_{\ell mn}(x')\right)^{\dagger}_{\dot\alpha}\right] \;=\; \frac{1}{\sin\theta}\delta^{(3)}(\vec{x}-\vec{x}\,')\,\mathbb{1}_{\alpha\dot\alpha} \;, \tag{A.24}$$

and

$$\int \mathrm{d}^3x\sqrt{g_3}\; S^{\pm}_{\ell mn\,\alpha}(x)\left(S^{\pm'}_{\ell'm'n'}(x)\right)^{\dagger}_{\dot\alpha}\mathbb{1}^{\alpha\dot\alpha} \;=\; \delta_{\ell,\ell'}\delta_{m,m'}\delta_{n,n'}\delta^{\pm,\pm'} \;, \tag{A.25}$$

where the integral is on the unit three-sphere. Using that

$$
\begin{aligned}
L_+ Y_\ell^{mn} &= \frac{1}{2}\sqrt{\ell(\ell+2)-4m(m+1)}\,Y_\ell^{m+1,n} \ , \\
L_- Y_\ell^{m+1,n} &= \frac{1}{2}\sqrt{\ell(\ell+2)-4m(m+1)}\,Y_\ell^{mn} \ ,
\end{aligned}
\tag{A.26}
$$

one can verify that

$$
\mathcal{O}_f S^\pm_{\ell mn} = \lambda^\pm_\ell S^\pm_{\ell mn} \ , \tag{A.27}
$$

with

$$
\lambda^+_\ell = -\frac{\alpha_f}{2}(\ell+2)+\gamma_f \ , \qquad \lambda^-_\ell = \frac{\alpha_f}{2}\ell+\gamma_f \ . \tag{A.28}
$$

When $\beta_f \neq 0$, the spinor spherical harmonics given by (A.21) are not eigenspinors of the operator $\mathcal{O}_f$, except the special cases

$$
\begin{aligned}
\mathbf{S}^{\text{special}+}_{\ell n} &\equiv S^-_{\ell,\frac{\ell}{2},n} = \begin{pmatrix} Y_\ell^{\frac{\ell}{2},n} \\ 0 \end{pmatrix} \ , \qquad \mathbf{S}^{\text{special}-}_{\ell n} \equiv S^-_{\ell,-\frac{\ell}{2}-1,n} = \begin{pmatrix} 0 \\ Y_\ell^{-\frac{\ell}{2},n} \end{pmatrix} \ , \\
\mathcal{O}_f \mathbf{S}^{\text{special}\pm}_{\ell n} &= \lambda^{\text{special}\pm}_\ell \mathbf{S}^{\text{special}\pm}_{\ell n} \ , \qquad \lambda^{\text{special}\pm}_\ell = \left(\frac{\alpha_f}{2}\ell \pm \beta_f + \gamma_f\right) \ .
\end{aligned}
\tag{A.29}
$$

For the generic harmonics, the eigenspinors of $\mathcal{O}_f$ for general β_f are obtained by an $SO(2)$ rotation,

$$
\begin{pmatrix} \mathbf{S}^+_{\ell mn} \\ \mathbf{S}^-_{\ell mn} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{pmatrix} \begin{pmatrix} S^+_{\ell mn} \\ S^-_{\ell mn} \end{pmatrix} \ . \tag{A.30}
$$

The rotation matrix is given by

$$
\begin{aligned}
\mathcal{R}_{12} &= \mathcal{R}_{11} \frac{(\frac{\alpha_f}{2}(\ell+2)+\lambda^+_{\ell m}-\beta_f-\gamma_f)}{(\frac{\alpha_f}{2}\ell-\lambda^+_{\ell m}+\beta_f+\gamma_f)} \frac{\cos\nu^+_{\ell m}}{\cos\nu^-_{\ell m}} \ , \\
\mathcal{R}_{21} &= \mathcal{R}_{22} \frac{(-\frac{\alpha_f}{2}\ell+\lambda^-_{\ell m}-\beta_f-\gamma_f)}{(-\frac{\alpha_f}{2}(\ell+2)-\lambda^-_{\ell m}+\beta_f+\gamma_f)} \frac{\cos\nu^-_{\ell m}}{\cos\nu^+_{\ell m}} \ ,
\end{aligned}
\tag{A.31}
$$

with

$$
\lambda^\pm_{\ell m} = -\frac{\alpha_f}{2}+\gamma_f \pm \sqrt{\frac{\alpha_f^2}{4}(\ell+1)^2+\alpha_f\beta_f(1+2m)+\beta_f^2} \ . \tag{A.32}
$$

Requiring the matrix to be $SO(2)$ fixes all the $\mathcal{R}_{ij}$, with a choice of overall sign fixed by requiring the matrix to be the identity matrix for $\beta_f = 0$. We then have

$$
\mathcal{O}_f \mathbf{S}^\pm_{\ell mn} = \lambda^\pm_{\ell m} \mathbf{S}^\pm_{\ell mn} \ , \tag{A.33}
$$

for $\ell \geq 1$, $-\frac{\ell}{2} \leq m \leq -\frac{\ell}{2}-1$, and $-\frac{\ell}{2} \leq m \leq \frac{\ell}{2}$.

Appendix B

Hurwitz zeta function

In this appendix we include the definition of the Hurwitz zeta function and some useful properties. This is defined as the analytic continuation to complex $s \neq 1$, of the following series

$$\zeta_H(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \ , \tag{B.1}$$

which is convergent for any $\mathrm{Re}(s) > 1$. Notice that

$$\zeta_H(s,1) = \zeta(s) \ , \tag{B.2}$$

corresponds to the Riemann zeta function.

For $s = -k$, where $k = 0, 1, 2, \ldots$, the Hurwitz zeta function reduces to the Bernoulli polynomials

$$\zeta_H(-k,a) = -\frac{B_{k+1}(a)}{k+1} \ , \tag{B.3}$$

defined as

$$B_k(a) = \sum_{n=0}^{k} \binom{k}{n} b_{k-n} a^n \ , \tag{B.4}$$

where b_n are the Bernoulli numbers. The first few ones read

$$\begin{aligned} B_0(a) &= 1 \ , \\ B_1(a) &= a - \frac{1}{2} \ , \\ B_2(a) &= a^2 - a + \frac{1}{6} \ , \\ B_3(a) &= a^3 - \frac{3}{2}a^2 + \frac{1}{2}a \ , \\ B_4(a) &= a^4 - 2a^3 + a^2 - \frac{1}{30} \ . \end{aligned} \tag{B.5}$$

The following formulas used in chapter 3 are easily proved,

$$\sum_{k=1}^{\infty} \frac{k}{(k+a)^s} \quad = \quad \zeta_H(s-1,a) - a\,\zeta_H(s,a)\ , \tag{B.6}$$

and

$$\sum_{k=1}^{\infty} \frac{k(k+1)}{(k+a)^s} \;=\; \zeta_H(s-2,a) + (1-2a)\zeta_H(s-1,a) + a(a-1)\zeta_H(s,a)\ . \tag{B.7}$$

Appendix C

Energy-momentum tensor and other currents of four-dimensional theories on curved backgrounds

In this appendix we provide explicit expressions for the energy-momentum tensor and other currents obtained from the (quadratic) chiral multiplet Lagrangian (3.12) from new minimal supergravity. Denoting with S the corresponding action, the energy-momentum tensor is defined as

$$T_{\mu\nu} \;=\; \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \ . \tag{C.1}$$

A straightforward but tedious computation yields

$$\begin{aligned} T_{\mu\nu} \;=\; & (2\delta^{\rho}_{(\mu}\delta^{\lambda}_{\nu)} - g_{\mu\nu}g^{\rho\lambda})\Big[D_\rho\widetilde{\phi}D_\lambda\phi + \frac{3}{2}rV_\rho V_\lambda\widetilde{\phi}\phi \\ & \qquad + (V_\rho + \kappa\,(\epsilon-1)\,K_\rho)\left(\mathrm{i}D_\lambda\widetilde{\phi}\,\phi - \mathrm{i}\widetilde{\phi}D_\lambda\phi\right)\Big] \\ & + \frac{r}{2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\widetilde{\phi}\phi + \frac{r}{2}\Big[g_{\mu\nu}\nabla_\rho\nabla^\rho(\widetilde{\phi}\phi) - \nabla_\mu\nabla_\nu(\widetilde{\phi}\phi)\Big] \\ & + \frac{\mathrm{i}}{2}D_{(\mu}\widetilde{\psi}\widetilde{\sigma}_{\nu)}\psi - \frac{\mathrm{i}}{2}\widetilde{\psi}\widetilde{\sigma}_{(\mu}D_{\nu)}\psi - \left(\frac{1}{2}V_{(\mu} + \kappa(1-\epsilon)K_{(\mu}\right)\widetilde{\psi}\widetilde{\sigma}_{\nu)}\psi \ , \end{aligned} \tag{C.2}$$

where the lower parenthesis denote symmetrization of the indices. Recall that we defined $D_\mu = \nabla_\mu - \mathrm{i}q_R A_\mu$, with q_R the R charges of the fields [26].

Below we collect some useful formulas for deriving this expression. For the bosonic part we used the variation of the Ricci tensor,

$$g^{\mu\nu}\delta R_{\mu\nu} \;=\; g_{\mu\nu}\nabla^\rho\nabla_\rho(\delta g^{\mu\nu}) - \nabla_\mu\nabla_\nu(\delta g^{\mu\nu}), \tag{C.3}$$

and we note that for any vector field X^μ,

$$[\nabla_\mu, \nabla_\nu]\, X^\mu \;=\; R_{\mu\nu} X^\mu \;. \tag{C.4}$$

For the femionic part, the variation of the action with respect to the metric gives

$$\delta S_{\rm fer}^{\rm chiral} \;=\; \int {\rm d}^4x \left[\delta\sqrt{-g}\,\mathcal{L}_{\rm fer}^{\rm chiral} + \sqrt{-g}\,\delta\mathcal{L}_{\rm fer}^{\rm chiral}\right] \;=\; \int {\rm d}^4x\sqrt{-g}\;\delta\mathcal{L}_{\rm fer}^{\rm chiral}\;, \tag{C.5}$$

where in the second equality we used that $\mathcal{L}_{\rm fer}^{\rm chiral}$ vanishes on-shell. The variation of the Lagrangian can be expressed in terms of variations of the vielbein and of the spin connection and reads

$$\delta\mathcal{L}_{\rm fer}^{\rm chiral} \;=\; \widetilde{\psi}\widetilde{\sigma}^A\left({\rm i}D_\mu + \frac{1}{2}V_\mu + \kappa(1-\epsilon)K_\mu\right)\psi\,\delta e_A{}^\mu - \frac{\rm i}{2}\widetilde{\psi}\widetilde{\sigma}^\mu\sigma^{AB}\psi\,\delta\omega_{\mu AB}\;, \tag{C.6}$$

where $A = 0,1,2,3$ is a frame index.

Using the property that the vielbein is covariantly constant,

$$0 \;=\; \nabla_\mu e_\nu{}^A \;=\; \partial_\mu e_\nu{}^A - \Gamma^\rho_{\mu\nu}e_\rho{}^A + \omega_\mu{}^A{}_B e_\nu{}^B\;, \tag{C.7}$$

we read off the variation of the spin connection

$$\delta\omega_{\mu AB} \;=\; \delta\Gamma^\rho_{\mu\nu}e_{A\rho}e_B{}^\nu - e_B{}^\nu\nabla_\mu(\delta e_{A\nu})\;. \tag{C.8}$$

Further, using the variation of the Christoffel symbol

$$\delta\Gamma^\sigma_{\mu\nu} \;=\; \frac{1}{2}g_{\mu\lambda}g_{\nu\rho}\nabla^\sigma(\delta g^{\lambda\rho}) - g_{\lambda(\mu}\nabla_{\nu)}(\delta g^{\lambda\sigma})\;, \tag{C.9}$$

and

$$\delta e_{A\nu} \;=\; -g_{\nu\beta}e_{A\alpha}\delta g^{\alpha\beta} + g_{\mu\nu}\delta e_A{}^\mu\;, \tag{C.10}$$

we can write the variation of the spin connection as

$$\delta\omega_{\mu AB} \;=\; \nabla_\nu\left(g_{\mu\lambda}e_{[A}{}^\nu e_{B]\rho}\delta g^{\lambda\rho} + \frac{1}{2}e_{A\lambda}e_{B\rho}\delta^\nu_\mu\delta g^{\lambda\rho} - e_{B\rho}\delta^\nu_\mu\delta e_A{}^\rho\right)\;. \tag{C.11}$$

Using this, the second term of (C.6) can be written as,

$$\begin{aligned}
-\frac{\rm i}{2}\widetilde{\psi}\widetilde{\sigma}^\mu\sigma^{AB}\psi\,\delta\omega_{\mu AB} \;=\;& -\widetilde{\psi}\widetilde{\sigma}^A\left({\rm i}D_\mu + \frac{1}{2}V_\mu + \kappa(1-\epsilon)K_\mu\right)\psi\,\delta e_A{}^\mu\\
&+\frac{\rm i}{4}\left[D_\mu\widetilde{\psi}\widetilde{\sigma}_\nu\psi - \widetilde{\psi}\widetilde{\sigma}_\mu D_\nu\psi\right]\delta g^{\mu\nu}\\
&-\frac{1}{2}\left(\frac{1}{2}V_\mu + \kappa(1-\epsilon)K_\mu\right)\widetilde{\psi}\widetilde{\sigma}_\nu\psi\,\delta g^{\mu\nu}\;,
\end{aligned} \tag{C.12}$$

up to a total divergence. Substituting this back into (C.6), the terms containing $\delta e_A{}^\mu$

cancel. The remaining terms are all proportional to $\delta g^{\mu\nu}$ and give the fermionic part of the energy-momentum tensor (C.2).

We also used the following identities for the σ-matrices in Lorentzian signature

$$\begin{aligned} \sigma^A\widetilde{\sigma}^B\sigma^C &= -\eta^{AB}\sigma^C + \eta^{AC}\sigma^B - \eta^{BC}\sigma^A + \mathrm{i}\epsilon^{ABCD}\sigma_D \ , \\ \widetilde{\sigma}^A\sigma^B\widetilde{\sigma}^C &= -\eta^{AB}\widetilde{\sigma}^C + \eta^{AC}\widetilde{\sigma}^B - \eta^{BC}\widetilde{\sigma}^A - \mathrm{i}\epsilon^{ABCD}\widetilde{\sigma}_D \ , \end{aligned} \tag{C.13}$$

with $\epsilon^{0123} = -1$, and the identities

$$[\nabla_\mu, \nabla_\nu]\,\psi \;=\; -\frac{1}{2}R_{\mu\nu AB}\sigma^{AB}\psi \ , \qquad [\nabla_\mu, \nabla_\nu]\,\widetilde{\psi} \;=\; -\frac{1}{2}R_{\mu\nu AB}\widetilde{\sigma}^{AB}\widetilde{\psi} \ , \tag{C.14}$$

valid for generic spinors ψ, $\widetilde{\psi}$.

One can easily compute the Ferrara-Zumino current

$$J^\mu_{\mathrm{FZ}} \;=\; -\frac{2}{3}\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta V_\mu} \ , \tag{C.15}$$

and the current

$$J^\mu_K = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta K_\mu} \ . \tag{C.16}$$

These read

$$J^\mu_{\mathrm{FZ}} \;=\; \frac{2}{3}\left(\mathrm{i}D^\mu\widetilde{\phi}\,\phi - \mathrm{i}\widetilde{\phi}D^\mu\phi + 3rV^\mu\widetilde{\phi}\phi - \frac{1}{2}\widetilde{\psi}\widetilde{\sigma}^\mu\psi\right) \ , \tag{C.17}$$

$$J^\mu_K \;=\; \kappa(1-\epsilon)\Big(\mathrm{i}D^\mu\widetilde{\phi}\,\phi - \mathrm{i}\widetilde{\phi}D^\mu\phi + \widetilde{\psi}\widetilde{\sigma}^\mu\psi\Big) \ , \tag{C.18}$$

and are not conserved. Starting with the expressions above, a further computation yields (3.73).

Appendix D

$SO(4)$-symmetric solutions of minimal gauged supergravity in five dimensions

In this appendix, we present an analysis of solutions to five-dimensional minimal gauged supergravity possessing $\mathbb{R} \times SO(4)$ symmetry. In particular, we prove that the only supersymmetry-preserving solution of this type is the singular one found long ago in [73]. To the knowledge of the authors of [4], where this proof of uniqueness was given, it had not previously appeared in the literature.

For simplicity, the notation of this appendix is independent of the rest of the thesis.

The most general ansatz for a metric and a gauge field with $\mathbb{R} \times SO(4)$ symmetry is

$$
\begin{aligned}
\mathrm{d}s^2 &= -U(r)\mathrm{d}t^2 + W(r)\mathrm{d}r^2 + 2X(r)\mathrm{d}t\,\mathrm{d}r + Y(r)\mathrm{d}\Omega_3^2 \ , && \text{(D.1)}\\
A &= A_t(r)\mathrm{d}t \ , && \text{(D.2)}
\end{aligned}
$$

where $\mathrm{d}\Omega_3^2$ is the metric on the round S^3 of unit radius,

$$
\begin{aligned}
\mathrm{d}\Omega_3^2 &= \frac{1}{4}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \ ,\\
\sigma_1 &= -\sin\psi\,\mathrm{d}\theta + \cos\psi\sin\theta\,\mathrm{d}\phi \ ,\\
\sigma_2 &= \cos\psi\,\mathrm{d}\theta + \sin\psi\sin\theta\,\mathrm{d}\phi \ ,\\
\sigma_3 &= \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi \ . && \text{(D.3)}
\end{aligned}
$$

The crossed term $X(r)\mathrm{d}t\mathrm{d}r$ in the metric can be removed by changing the t coordinate, so we continue assuming $X(r) = 0$. We will make use of the frame

$$
e^0 = \sqrt{U}\,\mathrm{d}t \ , \qquad e^{1,2,3} = \frac{1}{2}\sqrt{Y}\sigma_{1,2,3} \ , \qquad e^4 = \sqrt{W}\,\mathrm{d}r \ . \qquad \text{(D.4)}
$$

Equations of motion

We proceed by first solving the equations of motion and then examining the additional constraint imposed by supersymmetry. The action and equations of motion are given by equations (4.1) and (4.2). With the ansatz (D.2), the Maxwell equation is

$$0 \;=\; \nabla_\nu F^{\nu\mu} \qquad \Leftrightarrow \qquad 0 \;=\; A_t'' + \frac{1}{2} A_t' \left(\log \frac{Y^3}{UW} \right)' \; . \tag{D.5}$$

This can be integrated to

$$A_t' \;=\; c_1 \sqrt{\frac{UW}{Y^3}} \; , \tag{D.6}$$

with c_1 a constant of integration. The Einstein equations read (using frame indices)

$$\begin{aligned} R_{00} &= -R_{44} \;=\; 4g^2 + \frac{(A_t')^2}{3UW} \; , \\ R_{11} &= R_{22} \;=\; R_{33} \;=\; -4g^2 + \frac{(A_t')^2}{6UW} \; , \end{aligned} \tag{D.7}$$

where the Ricci tensor components are

$$\begin{aligned} R_{00} &= \frac{U''}{2UW} - \frac{U'W'}{4UW^2} + \frac{3U'Y'}{4UWY} - \frac{U'^2}{4U^2W} \; , \\ R_{11} &= R_{22} \;=\; R_{33} \;=\; -\frac{U'Y'}{4UWY} + \frac{W'Y'}{4W^2Y} - \frac{Y''}{2WY} - \frac{Y'^2}{4WY^2} + \frac{2}{Y} \; , \\ R_{44} &= -\frac{U''}{2UW} + \frac{U'W'}{4UW^2} + \frac{U'^2}{4U^2W} + \frac{3W'Y'}{4W^2Y} - \frac{3Y''}{2WY} + \frac{3Y'^2}{4WY^2} \; . \end{aligned} \tag{D.8}$$

To solve these, let us define

$$T(r) \;=\; U(r)W(r)Y(r) \; . \tag{D.9}$$

Combining two of the Einstein equations yields,

$$0 \;=\; R_{00} + R_{44} \;=\; \frac{3U}{4T^2} \left(T'Y' - 2TY'' \right) \; , \tag{D.10}$$

which can be integrated to

$$T(r) \;=\; c_2 \, Y'^2(r) \; , \tag{D.11}$$

with $c_2 \neq 0$ a constant of integration. Using this, the angular components of the Einstein equations can be integrated, yielding

$$U(r) \;=\; 4c_2 + 4c_2 g^2 Y + \frac{1}{Y} c_3 + \frac{c_1^2 c_2}{3Y^2} \; , \tag{D.12}$$

with a third constant of integration c_3. This solves all the equations of motion.

We can now use the freedom to redefine the radial coordinate to choose one of the functions. In particular, we can choose the function $W(r)$ so that $WU = 4s^2$, where we take $s > 0$. From (D.9) and (D.11) we then obtain

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}r}\right)^2 = \frac{4s^2}{c_2}Y \qquad \Rightarrow \qquad Y(r) = \frac{s^2}{c_2}r^2 \ , \tag{D.13}$$

where we used the freedom to shift r to set to zero an integration constant. Finally, after performing the trivial redefinitions $r^{\text{old}} = \frac{\sqrt{c_2}}{s}r^{\text{new}}$, $U^{\text{old}} = 4c_2U^{\text{new}}$, $t^{\text{new}} = 2\sqrt{c_2}t^{\text{old}}$, we arrive at the solution

$$\mathrm{d}s^2 = -U(r)\mathrm{d}t^2 + \frac{1}{U(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2 \ , \tag{D.14}$$

$$A = \left(c_4 - \frac{c_1}{2r^2}\right)\mathrm{d}t \ , \tag{D.15}$$

with

$$U(r) = 1 + g^2r^2 + \frac{c_3}{4c_2r^2} + \frac{c_1^2}{12r^4} \ , \tag{D.16}$$

and c_4 another arbitrary constant. Hence, the solution depends on three constants: c_1, which is essentially the charge, the ratio c_3/c_2, and c_4 which is quite trivial but may play a role in global considerations.

Supersymmetry

The integrability condition of the Killing spinor equation (4.3) is

$$\begin{aligned} 0 = \mathcal{I}_{\mu\nu}\epsilon \ \equiv\ & \frac{1}{4}R_{\mu\nu\kappa\lambda}\Gamma^{\kappa\lambda}\epsilon + \frac{\mathrm{i}}{4\sqrt{3}}\left(\Gamma_{[\mu}{}^{\kappa\lambda} + 4\Gamma^{\kappa}\delta^{\lambda}_{[\mu}\right)\nabla_{\nu]}F_{\kappa\lambda}\epsilon \\ & + \frac{1}{48}\left(F_{\kappa\lambda}F^{\kappa\lambda}\Gamma_{\mu\nu} + 4F_{\kappa\lambda}F^{\kappa}{}_{[\mu}\Gamma_{\nu]}{}^{\lambda} - 6F_{\mu\kappa}F_{\nu\lambda}\Gamma^{\kappa\lambda} + 4F_{\kappa\lambda}F_{\rho[\mu}\Gamma_{\nu]}{}^{\kappa\lambda\rho}\right)\epsilon \\ & + \frac{\mathrm{i}g}{4\sqrt{3}}\left(F^{\kappa\lambda}\Gamma_{\kappa\lambda\mu\nu} - 4F_{\kappa[\mu}\Gamma_{\nu]}{}^{\kappa} - 6F_{\mu\nu}\right)\epsilon + \frac{g^2}{2}\Gamma_{\mu\nu}\epsilon \ , \end{aligned} \tag{D.17}$$

where we used $[\nabla_\mu, \nabla_\nu]\epsilon = \frac{1}{4}R_{\mu\nu\kappa\lambda}\Gamma^{\kappa\lambda}\epsilon$. The Γ-matrices can be taken to be

$$\Gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \qquad \Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \Gamma_4 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{D.18}$$

with σ_i the three Pauli matrices.

A necessary condition for the solution to preserve supersymmetry is that

$$\det{}_{\text{Cliff}}\,\mathcal{I}_{\mu\nu} = 0 \qquad \text{for all } \mu,\nu \ , \tag{D.19}$$

where the determinant is taken over the spinor indices.For the $\mathbb{R}\times SO(4)$ invariant solution one finds (in flat indices a, b):

$$\det_{\text{Cliff}} \mathcal{I}_{ab} = \frac{9\,(16c_1^2c_2^2 - 3c_3^2)^2}{24^4 c_2^4\, r^{16}} \begin{pmatrix} 0 & 1 & 1 & 1 & 81 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 81 & 1 & 1 & 1 & 0 \end{pmatrix}_{ab} . \tag{D.20}$$

Hence, the supersymmetry condition is

$$\frac{c_3}{c_2} = -\frac{4}{\sqrt{3}} c_1 , \tag{D.21}$$

where we fixed a sign without loss of generality. Plugging this back into (D.16), we have

$$U(r) = \left(1 - \frac{c_1}{2\sqrt{3}\, r^2}\right)^2 + g^2 r^2 . \tag{D.22}$$

This recovers a solution first found in [73]. It is also obtained from (4.66)–(4.68) by setting $\alpha = 0$ and changing the radial coordinate.

Therefore we conclude that *in the context of minimal gauged supergravity, the most general supersymmetric solution possessing* $\mathbb{R} \times SO(4)$ *symmetry is the one-parameter family found in [73].* This preserves four supercharges and has a naked singularity.

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